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Path Interdependence in a Dynamic Two Country Heckscher-Ohlin Model

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Abstract

The closed economy neoclassical model predicts lung-run convergence in per-capita income. We show, within a neoclassical framework, that international trade among two countries differing only in their initial capital endowment generates long-run income differences. Our results suggests that trade creates opposite incentives to accumulate capital. Transitionally, the returns to investment with trade are smaller for countries initially less endowed with capital as when compared to their autarchic situation, while the reverse happens for those countries most endowed with capital. Thus, countries starting with relatively less (more) capital end, in the long run, with less (more) capital than in autarchy.

Key words: International trade, Development, Multiple Equilibria

JEL Classification: O41, F43, F11

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1. Introduction

The closed economy neoclassical version of the Ramsey model predicts that the level of capital stock to which a country converges is independent of its initial stock. Hence, countries that differ only in their initial levels of capital stock should converge to the same steady state and, consequently, share common levels of income in the long run. This prediction is based on the assumption of autarchy, and may not occur when countries engage in international trade. It is well known that the large discrepancies in income levels among countries can be attributed to the point in time at which countries started a sustained growth regime (Galor, 2005). A country's integration into world markets is often identified as an important contributor to its ability to sustain growth, although this point of view has its skeptics (Rodríguez and Rodrik, 2001). Others have attributed discrepancies in income levels among countries to a number of factors such as policies, savings rates and technology (Acemoglu and Ventura, 2002). Whereas the literature on endogenous growth has highlighted the importance of initial conditions on long-run capital and consumption levels (e.g. Lucas (1988) and Caballé and Santos (1993)), the theory of international trade (Stolper and Samuelson (1941)) suggests that if factors of production are not equally distributed among individuals, international trade will influence a country's distribution of income. Since factors of production are not equally distributed across countries, one might ask: will international trade alleviate or exacerbate income differences across countries in the long run? Others have posed similar questions. For example, Atkeson and Kehoe (2000) ask: How does the timing of a country's development relative to that of the rest of the world affect the path of a country's development?

Our paper is related to the work of Chen (1992), in that we focus on two countries that are not in long-run equilibrium using a Heckscher-Ohlin-like structure. He considers a two-country two-good dynamic Heckscher-Ohlin model. He includes a leisure-work decision choice in the utility function and ascertains that if consumers have relatively small discount factors, long-run income levels across countries will differ, depending on their initial conditions. Ventura (1997) focuses on conditional convergence among countries using a two-sector growth model in which intermediate products are traded internationally. He shows that convergence of per-capita output is affected by the elasticity of substitution between capital and labor. Acemoglu and Ventura (2002) consider a world economy consisting of a continuum of small countries that trade intermediate inputs internationally. Capital is employed in the production of intermediate products and, in turn, intermediates are used to produce two non-internationally traded commodities, investment and consumption goods. Countries differ in technologies, savings and economic policies. To avoid with specialization they employ the Armington specification. They demonstrate that rich countries are those that have low discount factors, create incentives to invest and have better technologies. Mountford (1998) employs an overlapping-generations model with two countries, and shows they converge to different income levels if consumers' time rates of discount vary across countries. Bajona and Kehoe (2006) analyze the discrete time version of Ventura's model and presume that consumption and investment are identical composites of two trade goods. Using a logarithmic utility function and assuming that capital fully depreciates in a single period, they demonstrate that initial conditions influence long-run per-capita income and that parameter values affect the distribution of income per-capita across countries over time.

Overall, our work more closely parallels that of Atkeson and Kehoe. They focus on a single country, in the presence of the rest of the world, that has converged to its long-run equilibrium. They find that the timing of a country's development, relative to the rest of the world, affects the path of that country's development. A country that has technologies and inter-temporal preferences as the rest of the world, but begins the development process with a capital-labor ratio lower than that of the rest of the world –a late bloomer–ends up with a permanently lower level of income than early-blooming countries. By not focusing on two or more countries that remain in transition to long-run equilibrium, Atkeson and Kehoe's analysis cannot draw inferences regarding differences in countries' transition paths, nor can they infer whether these paths are unique and converge to a unique steady state.

We consider a world economy of two countries, neither of which have converged to their long-run equilibrium. The countries produce two commodities, an investment good and a perishable consumption good. Consumers have identical preferences and identical discount factors, and derive satisfaction from consuming the perishable consumption good. Hence, countries only differ in their initial capital endowment. Restricting our analysis to the non-specialization case, we prove within the framework of a neoclassical model that different initial capital stock endowments are sufficient to generate long-run income differences across countries. We show that in state space the set of steady states is a ray, and demonstrate that, depending on initial conditions, economies converge to a point on this ray. In other words, different initial endowments of capital can lead to long-run differences in capital, consumption and income across countries. We find that while the steady-state world capital stock is unique, our two otherwise identical countries may converge to different levels of capital stock. We demonstrate that the early bloomer (i.e. the country with a higher initial level of capital) converges to a higher long-run level of capital stock than the late bloomer, and the early bloomer's level of capital is higher than that the level obtained in autarchy. The early bloomer enjoys a higher level of consumption and, if the capital-producing sector is capital intensive, its pattern of trade is to export the capital-intensive good and import the perishable consumption good. The savings rate of the early bloomer can be higher than that of the late bloomer—a difference which can persist in the long-run. These results suggest that the role played by initial capital stocks in many growth divergence/convergence studies is likely to be more complex, and that per-country capital endowments as a proportion of world capital may also be relevant to analyze growth divergence/

convergence across countries.

In the next section, the two country - two sector world economy is described, and key results are derived. We then restrict the parameters of the model so as to yield constant savings rates in each country to make more explicit and reinforce the results obtained in the previous section. Our model demonstrates that the late bloomer's returns to capital accumulation are always larger in autarchy than under international trade, whereas the early bloomer's returns to capital accumulation are always smaller in autarchy. With international trade, the price of the investment good is determined by the world's per-capita capital, and the rental price of capital must in turn be influenced by changes in the price of the investment good in a Stolper-Samuelson fashion. The late bloomer has a smaller per-capita capital than the world's per-capita capital, while the inverse holds for the early bloomer. International trade must therefore have an opposite effect on the returns to capital accumulation for early and late bloomers. Consequently, in transition, the late bloomer's terms of trade favor the non-capital intensive good, thus inducing this country to save less when compared to autarchy. The formal analysis suggests that international trade negatively (positively) influences the net returns to the capital of the late (early) bloomers, regardless of factor intensities. The early bloomer converges to a capital stock that is larger than would otherwise prevail in autarchy, since it has greater incentives to accumulate capital, while the late bloomer ends up with less capital than in autarchy. With trade, the early bloomer postpones consumption. In this way, consumption levels diverge in the long run.

To further confirm our analytical results and to provide insights into rough orders of magnitude, solutions are obtained to an empirical model for the case in which both countries remain within their cone of diversification. The results show transition paths corresponding to different country levels of initial capital stocks and corresponding different steady-states. Trade is shown to generate long-run differences of more than 35 percent in capital stocks across countries, depending upon differences in initial capital stocks.

2. The 2x2x2 economy.

The world economy consists of two countries, each of which employs capital and labor using constant elasticity of substitution technologies to produce a capital good Y_{xj} and a perishable consumption good Y_{cj} that are internationally traded. Each economy consists of a representative individual. Consumers' preferences are identical across countries, and are characterized by a constant elasticity of intertemporal substitution $\left(\theta^{-1}\right)$ utility function and discount future utility of consumption at rate $\rho > 0$ (also presumed to be identical across countries). At the beginning of time, each economy j is endowed with an identical amount of labor, L_j, but their endowment of capital $K_j(0)$, can differ. We call the late (early) bloomer the country that starts with a relatively smaller (larger) initial endowment of capital. To isolate the effect of different initial capital endowments, we do not consider technological change or population growth. Furthemore, the countries are not permitted to hold each other's assets.

2.1. Consumers' optimization problem.

The instantaneous utility of consumption c_j in the j-th country is given by

$$U(c_j) = \begin{cases} \frac{c_j^{1-\theta}}{1-\theta} \text{ if } \theta \neq 1 \\ \ln c_j \text{ if } \theta = 1 \end{cases} \tag{1}$$

where $\theta > 0$. The consumer of country j (for $j = 1, 2$) chooses paths of consumption $c_j(t)$, and assets $K_j(t)$ given prices to solve

$$\max \int_0^\infty U(c_j(t))\, e^{-\rho t} dt \tag{2}$$

subject to

$$\dot{K}_j(t) = \frac{\left(w_{Kj}(t) - \delta p(t)\right) K_j(t) + w_{Lj}(t) L_j - c_j(t)}{p(t)} \tag{3}$$

where the endowment of labor is normalized to unity ($L_j = 1$). The j-th country's rental price of capital and labor wage rate are denoted by $w_{Kj}(t)$ and $w_{Lj}(t)$, respectively, and δ is the constant and common rate of capital depreciation. Since the consumption and investment goods are internationally traded, their prices are equal across countries. The price of the investment good at instant t is denoted by $p(t)$, and the price of the consumption good is treated as the numéraire.

The first-order and transversality conditions of problem (2) are given by

$$c_j^{-\theta} = \frac{\lambda_j}{p}, \quad \lambda_j \left(\frac{w_{Kj}}{p} - \delta - \rho \right) = -\dot{\lambda}_j, \quad \lim_{t \to \infty} \lambda_j(t) K_j(t) e^{-\rho t} = 0 \quad \text{for } j = 1, 2 \tag{4}$$

where λ_j is the co-state variable associated with the constraint (3) (to avoid notational cluster, we omit expressing variables as a function of time unless needed for clarity). The Euler condition of the consumer of country j is therefore given by

$$\frac{\dot{c}_j}{c_j} = \frac{1}{\theta}\left(-\frac{\dot{\lambda}_j}{\lambda_j} + \frac{\dot{p}}{p} \right) = \frac{1}{\theta}\left(\frac{w_{Kj}}{p} + \frac{\dot{p}}{p} - \delta - \rho \right) \quad \text{for } j = 1, 2. \tag{5}$$

2.2. Firms

The technologies for producing the investment and consumption good are, respectively,

$$Y_{xj} = A\left(\alpha K_{xj}^{\gamma} + (1-\alpha) L_{xj}^{\gamma}\right)^{\frac{1}{\gamma}}, \qquad Y_{cj} = Q\left(\beta K_{cj}^{\gamma} + (1-\beta) L_{cj}^{\gamma}\right)^{\frac{1}{\gamma}} \tag{6}$$

where K_{ij} and L_{ij} denote the capital and labor services employed in the production of output $i = x, c$ in country $j = 1, 2$. Subscripts x and c denote the investment good and consumption good, respectively. $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha \neq \beta$ are distribution parameters. A and Q are positive constants and $-\infty < \gamma \leqq 0$, that is, we consider the case in which capital and labor are necessary for production. Let $\frac{1}{1-\gamma} \equiv \chi$ denote the elasticity of substitution between capital and labor.

We presume no factor intensity reversals and, therefore, assume γ to be equal across goods. Profit maximization in each sector implies

$$\begin{aligned} K_{xj} &= \left(\frac{\alpha A^{\gamma} p}{w_{Kj}}\right)^{\chi} Y_{xj}, & L_{xj} &= \left(\frac{(1-\alpha) A^{\gamma} p}{w_{Lj}}\right)^{\chi} Y_{xj}, \\ K_{cj} &= \left(\frac{\beta Q^{\gamma}}{w_{Kj}}\right)^{\chi} Y_{cj}, & L_{cj} &= \left(\frac{(1-\beta) Q^{\gamma}}{w_{Lj}}\right)^{\chi} Y_{cj} \end{aligned} \tag{7}$$

Using (6) and (7), we obtain the zero profit condition that output price equals marginal cost given by

$$p = \begin{cases} \frac{1}{A}\left(\frac{\alpha^{\chi}}{w_{Kj}^{\gamma\chi}} + \frac{(1-\alpha)^{\chi}}{w_{Lj}^{\gamma\chi}}\right)^{-\frac{1}{\gamma\chi}} & \text{for } \gamma < 0 \\ \bar{A}^{-1} w_{Kj}^{\alpha} w_{K_{Lj}}^{1-\alpha} & \text{for } \gamma = 0 \end{cases}, \qquad 1 = \begin{cases} \frac{1}{Q}\left(\frac{\beta^{\chi}}{w_{Kj}^{\gamma\chi}} + \frac{(1-\beta)^{\chi}}{w_{Lj}^{\gamma\chi}}\right)^{-\frac{1}{\gamma\chi}} & \text{for } \gamma < 0 \\ \bar{Q}^{-1} w_{Kj}^{\beta} w_{K_{Lj}}^{1-\beta} & \text{for } \gamma = 0 \end{cases} \tag{8}$$

where $\bar{A} = \alpha^{\alpha}(1-\alpha)^{1-\alpha} A$ and $\bar{Q} = \beta^{\beta}(1-\beta)^{1-\beta} Q$.

Definition.- An equilibrium are paths of quantities $c_j(t)$, $K_j(t)$, $Y_{ij}(t)$, $K_{ij}(t)$, $L_{ij}(t)$ and prices $w_{Kj}(t)$, $w_{Lj}(t)$ and $p(t)$, such that, given prices, $c_j(t)$ and $K_j(t)$ solve the optimization problem (2) of the consumer of country j for $j = 1, 2$. Given prices $K_{ij}(t)$ and $L_{ij}(t)$, for $i = x, c$ and $j = 1, 2$ solve the profit maximization problem of sector i in country j, and the following market clearing conditions are satisfied:

Market clearing for labor and capital in each country requires

$$L_{cj}(t) + L_{xj}(t) = L_j = 1 \tag{9}$$

$$K_{cj}(t) + K_{xj}(t) = K_j(t) \tag{10}$$

The international market clearing for the consumption good is given by

$$Y_{c1}(t) + Y_{c2}(t) = c_1(t) + c_2(t) = C(t) \tag{11}$$

where $C(t)$ denotes world consumption. Market clearing for the investment good equals

$$Y_{x1}(t) + Y_{x2}(t) = x_1(t) + x_2(t) \tag{12}$$

where $x_j(t)$ equals

$$x_j(t) = \dot{K}_j(t) - \delta K_j(t). \tag{13}$$

2.3. Solution

In the absence of specialization, factor price equalization occurs and thus $w_{Kj} = w_K$ and $w_{Lj} = w_L$. Equation (8) defines w_K and w_L in terms of the price of the investment good. More specifically,

$$w_K = \mathbf{W}_K(p) = \begin{cases} \Omega^{\frac{1}{\gamma\chi}} \left(\left(\frac{1-\beta}{A^\gamma p^\gamma} \right)^\chi - \left(\frac{1-\alpha}{Q^\gamma} \right)^\chi \right)^{-\frac{1}{\gamma\chi}} & \text{for } \gamma < 0 \\ \left(\bar{Q}^{-\frac{1}{1-\beta}} \left(p\bar{A} \right)^{\frac{1}{1-\alpha}} \right)^{\frac{(1-\beta)(1-\alpha)}{\alpha-\beta}} & \text{for } \gamma = 0 \end{cases} \tag{14}$$

$$w_L = \mathbf{W}_L(p) = \begin{cases} \Omega^{\frac{1}{\gamma\chi}} \left(\left(\frac{\alpha}{Q^\gamma} \right)^\chi - \left(\frac{\beta}{A^\gamma p^\gamma} \right)^\chi \right)^{-\frac{1}{\gamma\chi}} & \text{for } \gamma < 0 \\ \left(\bar{Q}^{-\frac{1}{\beta}} \left(p\bar{A} \right)^{\frac{1}{\alpha}} \right)^{\frac{\beta\alpha}{\beta-\alpha}} & \text{for } \gamma = 0 \end{cases} \tag{15}$$

where $\Omega \equiv \alpha^\chi (1-\beta)^\chi - \beta^\chi (1-\alpha)^\chi$. $\mathbf{W}_f(p)$ for $f = K, L$ are used to denote the relation between the rental price of factor f and the price of the investment good p. Note that the sign of Ω determines which sector uses capital (labor) intensively in its production. If $\Omega > (<) 0$, the investment good production uses capital (labor) intensively. As in the Heckscher-Ohlin model of international trade, if the investment (consumption) good uses capital intensively in its production and the price of the investment good increases (declines), then the rental price of capital increases and the labor wage rate declines.

Using the market clearing condition for labor and capital, we obtain the j-th country's supply of investment and consumption goods, respectively, given by

$$Y_{xj} = \frac{(1-\beta)^{\chi} w_K^{\chi} K_j - \beta^{\chi} w_L^{\chi} L_j}{A^{\gamma\chi} p^{\chi} \Omega}, \qquad Y_{cj} = \frac{\alpha^{\chi} w_L^{\chi} L_j - (1-\alpha)^{\chi} w_K^{\chi} K_j}{Q^{\gamma\chi} \Omega} \tag{16}$$

Market clearing in the consumption good implies

$$C = c_1 + c_2 = \frac{\alpha^{\chi} w_L^{\chi} (L_1 + L_2) - (1-\alpha)^{\chi} w_K^{\chi} (K_1 + K_2)}{Q^{\gamma\chi} \Omega} \tag{17}$$

which implicitly defines the price of the investment good p. Equation (17) and

$$\dot{c}_1 = \frac{1}{\theta}\left(\frac{w_K}{p} + \frac{\dot{p}}{p} - \delta - \rho\right) c_1, \tag{18}$$

$$\dot{c}_2 = \frac{1}{\theta}\left(\frac{w_K}{p} + \frac{\dot{p}}{p} - \delta - \rho\right) c_2, \tag{19}$$

$$\dot{K}_1 = \frac{(w_K - \delta p) K_1 + w_L L_1 - c_1}{p}, \tag{20}$$

$$\dot{K}_2 = \frac{(w_K - \delta p) K_2 + w_L L_2 - c_2}{p}, \tag{21}$$

form a system of four differential equations and one static equation in five variables c_1, c_2, K_1, K_2 and p, which together determine an equilibrium solution. Note that w_K, w_L can be substituted out employing $(14) - (15)$.

2.4. Steady states

If a steady state exists within each country's cone of diversification, the Euler condition of both countries implies

$$\frac{1}{\theta}\left(\frac{w_K^*}{p^*} - \delta - \rho\right) = 0 \quad \Rightarrow \quad \frac{w_K^*}{p^*} \equiv r^* = \delta + \rho \tag{22}$$

where the superscript $*$ denotes steady-state values. We denote the ratio $\frac{w_K^*}{p^*}$ as the rental rate of capital r^*. Using (8), and setting $p^* = \frac{w_K^*}{r^*}$, the steady-state rental price of capital and labor wage rate are, respectively, given by

$$w_K^* = \begin{cases} Q\left(\frac{(1-\beta)^{\chi}\left(\frac{r^*}{A}\right)^{\gamma\chi} - \Omega}{(1-\alpha)^{\chi}}\right)^{\frac{1}{\gamma\chi}} & \text{for } \gamma < 0 \\ \bar{Q}\left(\frac{\delta+\rho}{\bar{A}}\right)^{\frac{1-\beta}{1-\alpha}} & \text{for } \gamma = 0 \end{cases}, \quad w_L^* = \begin{cases} Q\left(\frac{(1-\beta)^{\chi}\left(\frac{r^*}{A}\right)^{\gamma\chi} - \Omega}{\left(\frac{r^*}{A}\right)^{\gamma\chi} - \alpha^{\chi}}\right)^{\frac{1}{\gamma\chi}} & \text{for } \gamma < 0 \\ \bar{Q}\left(\frac{\bar{A}}{\delta+\rho}\right)^{\frac{\beta}{1-\alpha}} & \text{for } \gamma = 0 \end{cases} \tag{23}$$

The price of the investment good at the steady state is

$$p^* = \frac{w_K^*}{r^*} \tag{24}$$

Setting (20) and (21) equal to zero, solving for c_1 and c_2, and substituting c_1 and c_2 into (17) gives the steady-state levels of capital in countries one and two, as the combinations of capital stocks K_1^* and K_2^* that satisfy the following equation

$$K_2^* = \frac{\alpha^\chi \left(w_L^*\right)^\chi - \Omega Q^{\gamma\chi} w_L^*}{(1-\alpha)^\chi \left(w_K^*\right)^\chi + \Omega Q^{\gamma\chi}\left(w_K^* - \delta p^*\right)} L - K_1^* \tag{25}$$

where L is the world labor supply $(L = L_1 + L_2 = 2)$. If these economies were closed, they would converge to the same steady state. In the presence of trade, values for K_1^* and K_2^* satisfying equation (25) determine a steady-state solution.

The steady-state level of consumption in each country equals the total factor returns,

$$c_j^* = \rho p^* K_j^* + w_L^* \text{ for } j = 1, 2 \tag{26}$$

Clearly, if $K_1^* \neq K_2^*$, then consumption across countries differs in the steady state.

The steady-state levels of production in each country are given by

$$Y_{xj}^* = \frac{(1-\beta)^\chi \left(w_K^*\right)^\chi K_j^* - \beta^\chi \left(w_L^*\right)^\chi L_j}{A^{\gamma\chi}\left(p^*\right)^\chi \Omega} \text{ for } j = 1, 2 \tag{27}$$

$$Y_{cj}^* = \frac{\alpha^\chi \left(w_L^*\right)^\chi L_j - (1-\alpha)^\chi \left(w_K^*\right)^\chi K_j^*}{Q^{\gamma\chi}\Omega} \text{ for } j = 1, 2 \tag{28}$$

Notice that only the combinations of (K_1^*, K_2^*) satisfying

$$\left(\frac{\alpha}{1-\alpha}\frac{w_L^*}{w_K^*}\right)^\chi > (<) \frac{K_j^*}{L_j} > (<) \left(\frac{\beta}{1-\beta}\frac{w_L^*}{w_K^*}\right)^\chi \text{ for } \alpha > (<)\, \beta \tag{29}$$

represent steady-state equilibria within the cone of diversification and that the combinations of (K_1^*, K_2^*) satisfying (25) are an infinite connected set. Next, we show that the segment of the ray (25) satisfying (29) is a self-attracting manifold.

3. Convergence

To facilitate the analysis of the model's convergence properties, the system is reduced to three differential equations. Using standard techniques, we show that the economies can converge to different steady states depending on their initial conditions.

3.1. Transforming and reducing the system

A difficulty one encounters when analyzing the asymptomatic properties of system (17) – (21) is that equations (18) and (19) contain the term $\frac{\dot{p}}{p}$. We, however, do not have direct information on this term. To analyze the asymptotic properties of the model it is useful to use the co-state variables and their differential equations obtained from (4) as follows[1]

$$\dot{\lambda}_1 = \left(\frac{-w_K}{p} + \delta + \rho\right)\lambda_1, \tag{30}$$

$$\dot{\lambda}_2 = \left(\frac{-w_K}{p} + \delta + \rho\right)\lambda_2, \tag{31}$$

$$\dot{K}_1 = \frac{(w_K - \delta p) K_1 + w_L L_1}{p} - \frac{1}{p}\left(\frac{p}{\lambda_1}\right)^{\frac{1}{\theta}}, \tag{32}$$

$$\dot{K}_2 = \frac{(w_K - \delta p) K_2 + w_L L_2}{p} - \frac{1}{p}\left(\frac{p}{\lambda_2}\right)^{\frac{1}{\theta}} \tag{33}$$

where p is implicitly defined by (17) and w_K and w_L can be substituted out using (14) – (15). Since the steady state is a manifold of dimension one, at least one of the eigenvalues of the Jacobian matrix of system (30) – (33) must be zero. Indeed, it is straightforward to demonstrate that the Jacobian matrix of (30) – (33) has a zero eigenvalue.

It is easiest to derive the convergence properties of the model by reducing the dimensionality of the dynamic system (30) – (33). Since the countries' Euler conditions are identical, the ratio of their consumption levels c_1/c_2 is constant throughout transition to the steady state. Let this ratio equal some constant $\mu > 0$. Using $c_2 = \left(\frac{p}{\lambda_2}\right)^{\frac{1}{\theta}}$, equation (4) implies

$$c_1 + c_2 = (\mu + 1)\, c_2 = (\mu + 1)\left(\frac{p}{\lambda_2}\right)^{\frac{1}{\theta}}. \tag{34}$$

Equation (17) can therefore be rewritten as

$$G(p, \lambda_2, K_1, K_2) = \left(\frac{\lambda_2}{p}\right)^{\frac{1}{\theta}}\left(\frac{\alpha^{\chi} w_L^{\chi}(L_1 + L_2) - (1-\alpha)^{\chi} w_K^{\chi}(K_1 + K_2)}{Q^{\gamma\chi}\Omega(\mu + 1)}\right) = 1 \tag{35}$$

Suppose $(p^o, \lambda_2^o, K_1^o, K_2^o)$ satisfies $G(p^o, \lambda_2^o, K_1^o, K_2^o) = 1$. Presuming differentiability, the Implicit Function Theorem implies the existence of a function $P(\lambda_2, K_1, K_2)$ defined on an open ball B about $(\lambda_2^o, K_1^o, K_2^o)$ such that

[1]Note from (4) that $c_j = \left(\frac{p}{\lambda_j}\right)^{\frac{1}{\theta}}$.

$$G\left(P\left(\lambda_2, K_1, K_2\right), \lambda_2, K_1, K_2\right) = 1 \text{ for all } \left(\lambda_2, K_1, K_2\right) \in B, \tag{36}$$

and

$$\frac{\partial P\left(\lambda_2^o, K_1^o, K_2^o\right)}{\partial q} = -\frac{G_q\left(p^o, \lambda_2^o, K_1^o, K_2^o\right)}{G_p\left(p^o, \lambda_2^o, K_1^o, K_2^o\right)} \quad \text{for} \quad q = \lambda_2, K_1, K_2 \tag{37}$$

where $G_q\left(p^o, \lambda_2^o, K_1^o, K_2^o\right)$ denotes the partial derivative of G with respect to q evaluated at *(*p^o, λ_2^o, K_1^o, K_2^o*)* satisfying $G\left(p^o, \lambda_2^o, K_1^o, K_2^o\right) = 1$. Using $\frac{c_1}{c_2} = \mu$ and $c_2 = \left(\frac{p}{\lambda_2}\right)^{\frac{1}{\theta}}$, we can therefore reduce the equilibrium conditions to a system of three differential equations in three *dynamic* variables λ_2, K_1, and K_2 as follows:

$$\begin{pmatrix} \dot{\lambda}_2 \\ \dot{K}_1 \\ \dot{K}_2 \end{pmatrix} = \begin{pmatrix} \left(-w_K/P\left(\lambda_2, K_1, K_2\right) + \delta + \rho\right)\lambda_2 \\ \left(\left(w_K - \delta p\right)K_1 + w_L L_1\right)/P\left(\lambda_2, K_1, K_2\right) - \mu\left(P\left(\lambda_2, K_1, K_2\right)^{1-\theta}/\lambda_2\right)^{\frac{1}{\theta}} \\ \left(\left(w_K - \delta p\right)K_2 + w_L L_2\right)/P\left(\lambda_2, K_1, K_2\right) - \left(P\left(\lambda_2, K_1, K_2\right)^{1-\theta}/\lambda_2\right)^{\frac{1}{\theta}} \end{pmatrix} \tag{38}$$

where $p = P\left(\lambda_2, K_1, K_2\right)$ is implicitly defined by (35), and as before w_K, w_L can be replaced by using $(14) - (15)$.

3.2. Convergence properties

From Li et al. (2003), we know that the system (38), a system of a lower dimension, preserves the dynamic properties of the original system (30) – (33). Note that any combination of the state variables (K_1, K_2) satisfying (25) and (29) constitutes a steady-state equilibrium. We now demonstrate that the segment of the ray (25) satisfying (29) is saddle path stable. This implies that different initial conditions of the state variables $\left(K_1(0), K_2(0)\right)$ may lead to a different steady state.

These results lead to two claims.

Claim 1. *Different initial conditions of the state variables K_1 and K_2* ***can*** *asymptotically lead to different steady-state values of* K_1^* and K_2^* *whose sum satisfies a unique world value* $K^* = K_1^* + K_2^*$.

Claim 2. *At each steady state* $\left(K_1^*, K_2^*\right)$ *there is a neighborhood containing a one-dimensional manifold convergent to a point on the ray of steady states at* $\left(K_1^*, K_2^*\right)$ *such that for all initial conditions* $\left(K_1(0), K_2(0)\right)$ *of this one-dimensional manifold, the equilibrium path converges exponentially to* $\left(K_1^*, K_2^*\right)$.

Proposition 1. (Convergence) *The Jacobian matrix of* (38) *evaluated at a steady state has a negative eigenvalue and two positive eigenvalues.*

Proof. See Appendix■

Since the set of steady states is a ray of dimension one (25) and the Jacobian matrix of (38) has a negative eigenvalue and two positive eigenvalues, from Li et al. (2003) it follows that claims 1 and 2 hold. Provided initial conditions are in the neighborhood of a steady states it also follows that for each steady state $(K_1^*,\ K_2^*)$ there is a unique path with different initial conditions converging to $(K_1^*,\ K_2^*)$. Note that a steady state is not stable in the sense that if we disturb this steady state by providing more capital to one of the economies, the countries converge to a different steady state.

Corollary 1. Countries that start with different initial endowments of capital end up with different income levels.

Proposition 2 *The gap between the consumption shares of the two countries at any point in time t is determined by the difference in initial capital endowments.*

Proof. Since consumption grows at equal rates across countries, the share of per-country consumption in aggregate consumption, $C(t) = c_1(t) + c_2(t)$, is constant,

$$\phi_j = \frac{c_j(t)}{C(t)} \tag{39}$$

Integrating each country's budget constraint and employing the transversality condition, we obtain

$$\phi_1 - \phi_2 = \frac{p(0)(K_1(0) - K_2(0))}{C(0)^\theta \lim\limits_{t\to\infty} \int_0^t C(\tau)^{1-\theta} e^{-\rho\tau} d\tau} \blacksquare \tag{40}$$

If the utility function is logarithmic, $U(c_j) = \ln c_j$, the difference in consumption across countries at time zero equals

$$c_1(0) - c_2(0) = \rho p(0)(K_1(0) - K_2(0)) \tag{41}$$

Another implication of equation (41) is that the more impatient countries are, (ρ large), the larger the consumption of the country with the largest initial capital stock. Thus, if $K_1(0) \neq K_2(0)$, consumption across countries will differ transitionally and at the steady state. The country that starts out with the largest capital endowment will forever enjoy higher consumption levels. The consumption ratio of country one and two is given by

$$\frac{\phi_1}{\phi_2} = \mu = \frac{K_1(0)\, C(0)^{-\theta}\, p(0) + \int_0^\infty C(\tau)^{-\theta}\, w_L(\tau)\, e^{-\rho\tau} d\tau}{K_2(0)\, C(0)^{-\theta}\, p(0) + \int_0^\infty C(\tau)^{-\theta}\, w_L(\tau)\, e^{-\rho\tau} d\tau} \tag{42}$$

Corollary 2. With international trade, the country that starts with the smallest initial capital endowment will have a steady-state consumption level lower than its steady-state autarchy consumption level.

Proposition 3 *The gap between the level of* countries' *capital stock in the steady state is determined by the difference in initial capital endowments.*

Proof. Substituting (26) into (40) we obtain

$$K_1^* - K_2^* = \frac{p(0)}{\rho p^*} \frac{C^*}{C(0)^\theta} \left(\frac{K_1(0) - K_2(0)}{\int_0^\infty C(\tau)^{1-\theta} e^{-\rho\tau} d\tau} \right) \tag{43}$$

where C^* denotes world consumption at the steady state, equalling

$$C^* = \frac{(1-\alpha)^\chi (w_K^*)^\chi \frac{(w_L^*)}{p^*} + \alpha^\chi (w_L^*)^\chi \rho}{(1-\alpha)^\chi \frac{(w_K^*)^\chi}{(p^*)} + Q^{\gamma\chi}\Omega\rho} L \blacksquare \tag{44}$$

It follows that the country with the largest initial endowment of capital converges to a steady-state capital stock that is larger than that of the other country. Compared to the steady-state level of capital in autarchy, with trade the country that started out with the largest initial endowment of capital will surpass the steady-state autarchy level of capital, while the other country asymptotically converges to a capital stock that is lower than the level it obtains in autarchy.

The differences in country capital stocks imply differences in the pattern of production. From (16) it is straightforward to verify that if production of the investment good is relatively capital intensive, $\alpha > \beta$, then the country with the largest initial stock of capital produces larger amounts of the investment good relative to the other country, while the other produces a larger amount of the consumption good.

Next we look at an analytical solution to the model under a restriction in parameter values[2]. In particular, this restriction in parameter values for the closed economy two-sector Ramsey model leads to the two-sector model by Uzawa (1963) of constant savings rates (as in the Solow model (1956)).

3.3. Analytical solution

Consider a particular solution of the model presented in Section 2.

Proposition 4. *If*

$$\rho = \delta(\alpha + \beta(\theta - 1)) - \delta, \qquad \alpha + \beta(\theta - 1) > 1, \qquad \gamma = 0 \tag{45}$$

then the world has a constant savings rate from the world income equal to $s = \frac{1}{\theta}$[3].

[2] $\rho = \delta(\alpha + \beta(\theta - 1)) - \delta$, $\gamma = 0$.

[3] Note that $\alpha + \beta(\theta - 1) > 1$ requires $\theta > 1$.

Proof. See Appendix■

Proposition 5. *Let* $K(0)$ *denote the world's initial capital endowment* *(*$K(0) = K_2(0) + K_1(0)$*), and let* $\kappa_j(0) = K_j(0)/K(0)$. *For* (45), *the share of each country's consumption in total consumption is given by*

$$\phi_j = \left(\beta - \frac{1-\alpha}{\theta - 1}\right)\left(\kappa_j(0) - \frac{L_j}{L}\right) + \frac{L_j}{L} \tag{46}$$

$$= \left(\frac{\rho}{\delta}\frac{1}{\theta-1}\right)\left(\kappa_j(0) - \frac{1}{2}\right) + \frac{1}{2}$$

Proof. See Appendix■

Since $\theta > 1$ must hold under (45), the term $\left(\frac{\rho}{\delta}\frac{1}{\theta-1}\right)$ is positive. Since we have assumed that each country has the same endowment of labor, $\frac{L_j}{L} = \frac{1}{2}$, it follows from (46) that the country with the largest proportion of initial capital ($\kappa_j(0) > 1/2$) will have a share in aggregate consumption of at least 1/2. This country then benefits proportionally more than the other country from the larger initial capital endowment, and the relative benefit depends upon the magnitude of $\frac{\rho}{\delta}\frac{1}{\theta-1} > 0$. Not surprisingly, the more consumers discount future utility of consumption (large ρ) and the more consumers are willing to intertemporally substitute consumption measured by θ^{-1}, the greater the benefit from having a larger initial endowment of capital. This benefit, however, is negatively affected by the size of the rate of capital depreciation (δ).

Proposition 6. *If* (45) *holds, transitionally the net returns to capital accumulation (*$(w_K + \dot{p})/p - \delta$*) with international trade are:*

i) lower for the late bloomer, and

ii) larger for the early bloomer

when compared to their respective autarchy case, regardless of sectoral factor intensities.

Proof. See Appendix■

Thus, an early bloomer's stock of capital is larger in the long run than in autarchy, since it has greater incentives to accumulate capital, while the late bloomer ends up with less capital than in autarchy. Since the early bloomer has greater incentives to save under trade than under autarchy, with trade it postpones consumption. Since the early bloomer postpones consumption and tends to save in the short run, in the long run it enjoys higher consumption levels than in autarchy, while the inverse happens for the late bloomer. In this way, consumption levels diverge in the long run.

Proposition 7. *If* (45) *holds, transitionally the labor wage rate* w_L *is:*

i) larger for the late bloomer, and

ii) lower for the early bloomer

when compared to their respective autarchy case, regardless of sectoral factor intensities.

Proof. See Appendix$\blacksquare$

Thus, the results of the standard Heckscher-Ohlin model of international trade also hold. In particular, if factors of production are not equally distributed across a country's individuals, international trade has income distribution effects.

The restriction (45) also implies other regularities in the evolution a country's share of capital stock and savings.

Proposition 8. *Given* (45) *, the ratio* $\kappa_j(t) = K_j(t)/K(t)$ *is constant for all* t *and*

$$\kappa_j = \frac{K_j(t)}{K(t)} = \frac{K_j(0)}{K(0)} \tag{47}$$

Proof. See Appendix$\blacksquare$

The restriction (45) implies a constant world saving rate, but not necessarily identical saving rates across countries. Let $s_j(t)$ denote the share of savings from income for the j-th country.

Proposition 9. *Given* (45) *, the* $j-th$ *country has a constant savings rate and moreover, if* $K_1(0) \neq K_2(0)$ *then*

$$s_1 \neq s_2. \tag{48}$$

and the rate s_j *is given by*

$$s_j = \frac{\delta}{\left[\delta\left((1-\alpha)+(1-\beta)(\theta-1)\right)\right]\left(\frac{L_j}{L}\frac{K(0)}{K_j(0)}\right)+\delta+\rho} \quad \text{for } j=1,2 \tag{49}$$

Proof. See Appendix$\blacksquare$

Consequently, (49) indicates that the country with the larger initial capital stock will also have a higher saving rate[4]. Whereas the capital stock of country j positively affects its own saving rate, the other country's capital endowment negatively affects country j's saving rate. This result contrast other literature showing that different savings rates are the result of differences in discount factors across countries (Mountford, 1998, Acemoglu and Ventura, 2002).

4. Some simulations

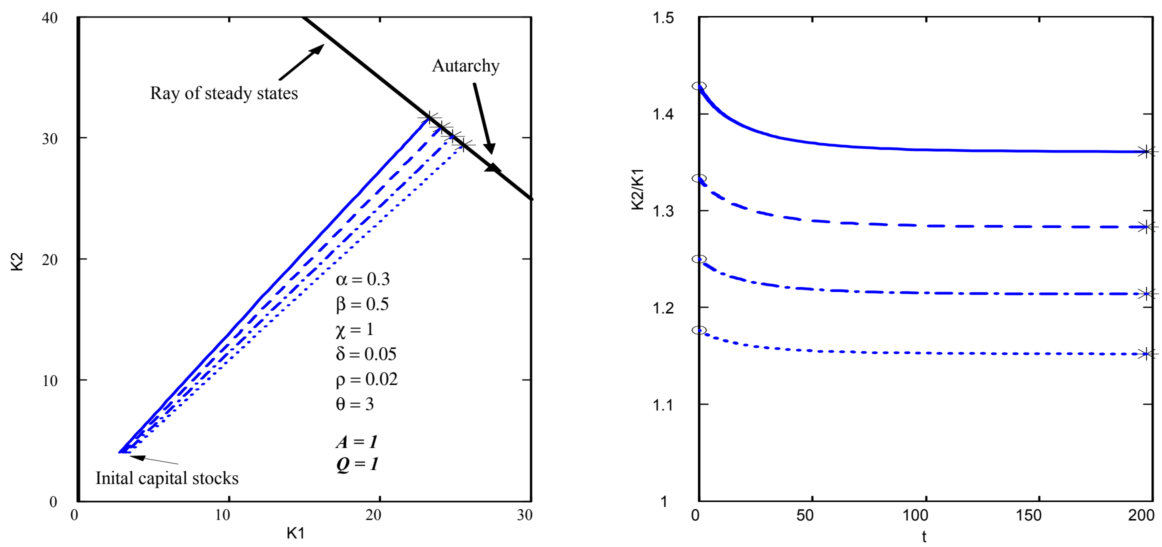
For illustrative purposes, we numerically solve the model, assuming the following parameter values:

$$\delta = .05, \quad \rho = .02, \quad \theta = 3, \quad \alpha = 0.5, \quad \beta = 0.3, \quad \gamma = 0, \quad A, Q = 1$$

[4] $\frac{\partial s_j}{\partial K_j(0)}\frac{K_j(0)}{s_j} = \frac{s_j}{\delta}\left[\delta\left((1-\alpha)+(1-\beta)(\theta-1)\right)\right]\frac{L_j}{L}\left(\frac{K(0)}{K_j(0)}-1\right) > 0$

To simulate the model we employ the relaxation algorithm for solving continuous time, multi-state, infinite-horizon differential systems developed by Trimborn et al. (2006).The left plot of Figure 1 displays a phase diagram of the countries' capital stocks. We consider four different cases in which the initial capital stock of country one differs (the horizontal axis), keeping the initial capital endowment of country two (the vertical axis) equal across simulations. The solid line is used to identify the base simulation, where the capital stock of country one equals 3.4 and the initial capital stock of country two equals 4. The other three simulations differ by subsequent decreases in the initial capital stock of country one by about six percent less than the previous simulation or subsequent decreases of 0.2 units of capital compared to the base simulation. Steady-state values are denoted by *. For comparison purposes, the triangle (▲) shows the autarchy level of steady-state capital stocks. We have also plotted the ray of steady states.

Figure 1. Phase diagram and ratio between capital stocks

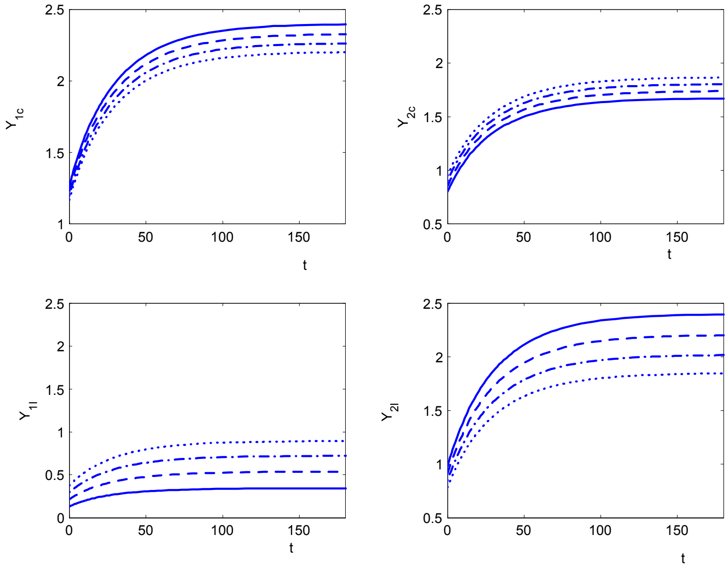


——— $(K_1(0),\ K_2(0)) = (3.4,\ 4.0)$ - - - $(K_1(0),\ K_2(0)) = (3.2,\ 4.0)$

- · - · $(K_1(0),\ K_2(0)) = (3.0,\ 4.0)$ · · · · $(K_1(0),\ K_2(0)) = (2.8,\ 4.0)$

Figure 1 confirms claim one that the economies converge to different steady states depending on initial conditions. In particular, the country that starts with the largest capital stock also converges to a capital stock that is larger than that of the other country. The country that starts with the largest capital stock also converges to a capital stock that is larger than the capital stock obtained in autarchy, while the opposite happens for the other country. The diagram on the right plots the evolution of the ratio of country two's capital stock to country one's capital stock. Countries remain within their cone of diversification so that specialization does not occur (see Figure 2). The right diagram in Figure 1 shows that the model with trade can generate long-run differences of more than 35 percent in capital stocks across countries.

Figure 2. Production Y_{1c}, Y_{1x}, Y_{2c} and Y_{2x}

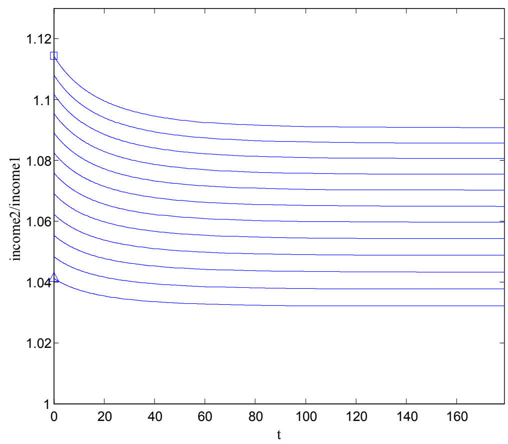


——— $(K_1(0),\ K_2(0)) = (3.4,\ 4.0)$ - - - $(K_1(0),\ K_2(0)) = (3.2,\ 4.0)$

- · - · $(K_1(0),\ K_2(0)) = (3.0,\ 4.0)$ · · · · $(K_1(0),\ K_2(0)) = (2.8,\ 4.0)$

Figure 3 plots income ratios for the four simulations performed. Long-run income differences range from appoximately 5 to 11 percent.

Figure 3. Income ratio $((Y_{2c}(t) + p(t)\,Y_{2x}(t)) \,/\, (Y_{1c}(t) + p(t)\,Y_{1x}(t))$

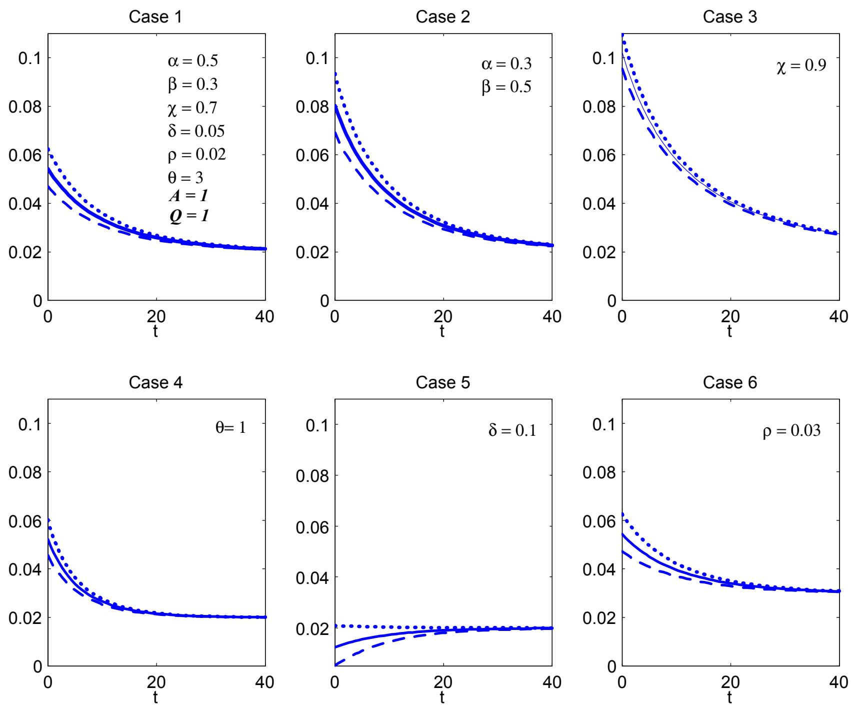


——— $(K_1(0),\ K_2(0)) = (3.4,\ 4.0)$ - - - $(K_1(0),\ K_2(0)) = (3.2,\ 4.0)$

- · - · $(K_1(0),\ K_2(0)) = (3.0,\ 4.0)$ · · · · $(K_1(0),\ K_2(0)) = (2.8,\ 4.0)$

When simulating the model, the possible set of parameter values to choose from is very large because estimates of parameter values vary considerably across studies[5]. Nonetheless, we want to test the sensitivity of the analytical results presented in section three to different parameter values. Figure 4 presents the simulation results for the net returns to capital under autarchy and under trade, assuming that the initial capital stocks of country one (the late bloomer) and country two (the early bloomer) equal 3.4 and 4, respectively. The parameter values we have chosen are indicated in the top-left graph (Case 1) in Figure 4. The other graphs in Figure 4 (Cases 2 to 6) are simulations with different parameter values from those presented in Case 1. The different parameter values from those of Case 1 are indicated in each plot. For example, Case 3 considers the elasticity substitution between labor and capital to be equal to $\chi = 0.9$, while the remaining parameters are as indicated in Case 1.

Figure 4. Net returns to capital $\left(\frac{w_K(t)}{p(t)} + \frac{\dot{p}(t)}{p(t)} - \delta\right)$



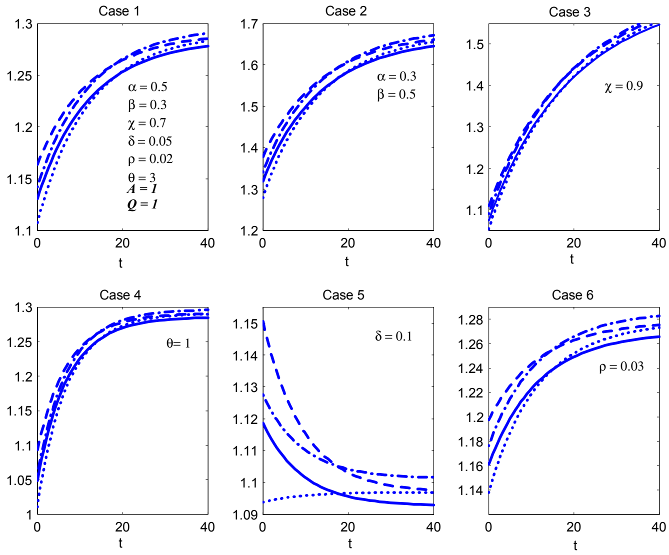
... Late bloomer autarchy — - - Early bloomer autarchy — —— Trade

Figure 4 confirms our analytical results that, regardless of factor intensities, the net returns to capital with international trade are transitionally smaller for the late bloomer, and larger for the early bloomer compared to their autarchy case. Thus, when compared to autarchy, in the short

[5] For example, estimates of the elasticity of intertemporal substitution (θ^{-1}) vary from approximately 0.06 to 2 (see for example Hall (1988) and Mulligan (2002)).

run, with trade the early bloomer has larger incentives to save while the opposite happens for the late bloomer. With trade, therefore, the early bloomer postpones consumption. Whereas in the short run, the consumption levels of the early bloomer are smaller under trade than in autarchy, the reverse happens for the late bloomer (see Figure 5). This implies that in the short run the consumption levels of the two countries are more similar (tend to converge) with trade. Since the early bloomer postpones consumption, in the long run it consumes more than in its autarchic situation. Thus, while in the short run consumption levels tend to converge with trade (compared to the autarchy situation), with trade long-run consumption levels diverge (see Figure 5).

Figure 5. Consumption under trade and autarchy

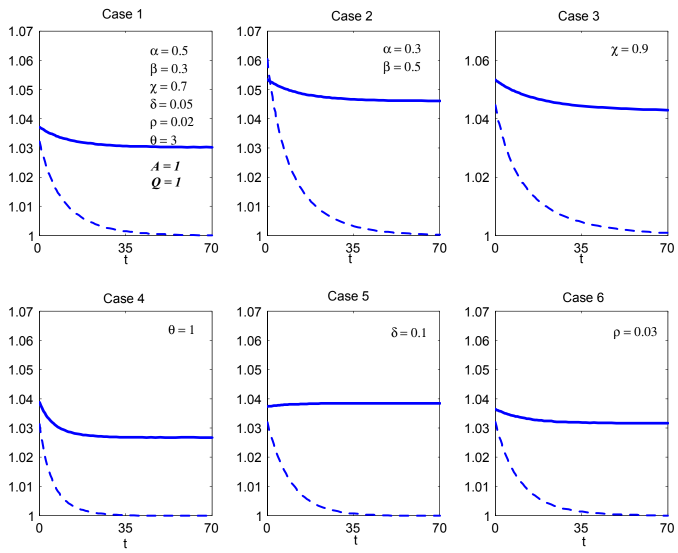


· · ·Late bloomer autarchy - - - Early bloomer autarchy

——— Late bloomer Trade - · - ·Early bloomer trade

In Figure 6 we present the ratio of country two's income to country one's income. Interestingly, when the production of the investment good is capital intensive, in the short run international trade (the solid line) widens the income gap between the two countries, compared to autarchy (the dashed line). Case two is the exception, which corresponds to the case when consumption uses capital intensively in its production. These simulations indicate that policy, in particular international trade policy, has a very distinct effect on consumption and income convergence (a situation in which consumption or income are more similar across countries). In the cases considered here, international trade always yields consumption convergence in the short run. When capital is used intensively in the production of the investment good international trade generates income divergence across countries not only in the long run also even more so in the short run.

Figure 6. Income ratios $\frac{Y_{2c}(t)+p(t)Y_{2x}(t)}{Y_{1c}(t)+p(t)Y_{1x}(t)}$ under autarchy and trade



5. Conclusions

Large discrepancies in income levels among countries can be attributed, in part, to the conditions at the starting point of the growth process. Those who start the process later, the late-bloomers, are typically characterized as having lower capital to output ratios than the early bloomers. The closed economy neoclassical version of the Ramsey model predicts that a county's long-run equilibrium is independent of its initial capital. If countries are integrated, according to our findings, they may not converge to similar capital and per-capita income levels in the long run. We consider a world of two open and competitive economies that produce an investment good and a perishable good. The economies only differ in the level of their initial capital stock. The early bloomer (the country whose initial capital stock exceeds that of the other country) is shown to converge not only to a higher level of capital than the late bloomer, but also to an amount that exceeds its autarchy level. In particular, we show that in state space, the set of steady states is a ray and demonstrate that, depending on initial conditions, the economies converge to a point on this ray. In other words, different initial endowments of capital can lead to long-run differences in capital, consumption and income across countries. These and corollary results are shown for the general case and, more specifically, for the case in which the model's parameters are restricted to allow a constant savings rate.

Our explanation of why international trade generates consumption and capital divergence in the long run is as follows. Since the late bloomer has less capital than the early bloomer, in autarchy the late bloomer's returns to capital must be larger than the early bloomer's returns to capital. With international trade, the price of the investment good must be determined by the world's per-capita capital, and the rental price of capital must in turn be influenced by changes in the price of the investment good in a Stolper-Samuelson fashion. Since the late bloomer has a smaller per-capita capital than the world's per-capita capital while the inverse holds for the early bloomer, international trade must have an opposite effect on the returns to capital accumulation for early and late bloomers. In particular, international trade must negatively (positively) influence the net returns to the capital of late (early) bloomers. With trade, the early bloomer converges to a capital stock that is larger than would otherwise prevail in autarchy, since it has greater incentives to accumulate capital, while the late bloomer ends with less capital than in autarchy. With trade, the early bloomer postpones consumption. In this way, consumption levels diverge in the long run.

Future research may deal with the issue of whether or not allowing for international borrowing and lending leads to equalization of income across countries.

6. Appendix

6.1. *Proof of* ***proposition 1 (Convergence)***

The Jacobian matrix of (38) equals

$$J^* = \begin{pmatrix} \Lambda \left.\frac{\partial P}{\partial \lambda_2}\right|_* & \Lambda \left.\frac{\partial P}{\partial K_1}\right|_* & \Lambda \left.\frac{\partial P}{\partial K_2}\right|_* \\ H_1 \left.\frac{\partial P}{\partial \lambda_2}\right|_* + \frac{1}{\theta}\frac{1}{p^*}\frac{c_1^*}{\lambda_2^*} & H_1 \left.\frac{\partial P}{\partial K_1}\right|_* + \rho & H_1 \left.\frac{\partial P}{\partial K_2}\right|_* \\ H_2 \left.\frac{\partial P}{\partial \lambda_2}\right|_* + \frac{1}{\theta}\frac{1}{p^*}\frac{c_2^*}{\lambda_2^*} & H_2 \left.\frac{\partial P}{\partial K_1}\right|_* & H_2 \left.\frac{\partial P}{\partial K_2}\right|_* + \rho \end{pmatrix} \tag{50}$$

where $\left.\frac{\partial P}{\partial q}\right|_*$ denotes the derivative of P with respect to $q = \lambda_2, K_1, K_2$ evaluated at a steady state and

$$\begin{aligned} \Lambda &= \left(w_K^* - \left.\frac{dW_K(p)}{dp}\right|_{p=p^*} p^* \right) \frac{\lambda_2^*}{(p^*)^2} = \left(1 - \left.\frac{dW_K(p)}{dp}\right|_{p=p^*} \frac{p^*}{w_K^*} \right) \frac{w_K^* \lambda_2^*}{(p^*)^2} \\ &= \left(\frac{\Omega}{w_K^{\gamma\chi}} - \left(\frac{1-\beta}{A^\gamma p^\gamma} \right)^\chi \right) \frac{w_K^{\gamma\chi}}{\Omega} \frac{w_K^* \lambda_2^*}{(p^*)^2} \text{ using (14) and factoring } \frac{w_K^{\gamma\chi}}{\Omega} \\ &= -\left(\frac{1-\alpha}{Q^\gamma} \right)^\chi \frac{w_K^{\gamma\chi}}{\Omega} \frac{w_K^* \lambda_2^*}{(p^*)^2} \text{ solving for } -\left(\frac{1-\beta}{A^\gamma p^\gamma} \right)^\chi \text{ from (14)} \end{aligned} \tag{51}$$

where $W_K(p)$ is the relation between the rental price of capital and the price of the investment good, as indicated in equation (14).

$$H_j = \left(\left.\frac{dW_K}{dp}\right|_{p=p^*} - \delta \right) \frac{K_j}{p^*} + \left.\frac{dW_L}{dp}\right|_{p=p^*} \frac{L_j}{p^*} \tag{52}$$

$$-\frac{(w_K^* - \delta p^*) K_j^* + w_L^* L_j}{(p^*)^2} - \frac{1-\theta}{\theta} \frac{c_j^*}{(p^*)^2} \text{ for } j = 1, 2$$

envelope properties of aggregate output of country j $(Y_{cj} + pY_{xj})$ imply[6]

$$H_j = -\frac{1}{(p^*)^2} \left(Y_{cj}^* + \frac{1-\theta}{\theta} c_j^* \right) \text{ for } j = 1, 2 \tag{53}$$

and the derivatives of $P(\lambda_2, K_1, K_2)$ are given by (37). Note that since $\frac{\partial P}{\partial K_1} = \frac{\partial P}{\partial K_2}$, the Jacobian matrix (50) can be rewritten as

$$J^* = \begin{pmatrix} J_{11}^* & J_{12}^* & J_{12}^* \\ J_{21}^* & J_{23}^* + \rho & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{32}^* + \rho \end{pmatrix} \tag{54}$$

where J_{ih}^* is the row i and column h element of matrix (50). One can readily verify that any matrix like that indicated in (54) has an eigenvalue equal to ρ. Let ε be an eigenvalue of matrix (54). The characteristic equation of (54) equals

$$(J_{11}^* - \varepsilon) \left[(J_{23}^* + \rho - \varepsilon) (J_{32}^* + \rho - \varepsilon) - J_{23}^* J_{32}^* \right] \tag{55}$$

$$-\underbrace{J_{12}^* J_{21}^* (J_{32}^*}_{a} + \rho - \varepsilon) + \underbrace{J_{12}^* J_{23}^* J_{31}^*}_{b} + \underbrace{J_{12}^* J_{21}^* J_{32}^*}_{a'} - \underbrace{J_{12}^* J_{31}^* (J_{23}^*}_{b'} + \rho - \varepsilon)$$

$$= (J_{11}^* - \varepsilon) \left[(J_{23}^* + \rho - \varepsilon) (J_{32}^* + \rho - \varepsilon) - J_{23}^* J_{32}^* \right] - J_{12}^* J_{21}^* (\rho - \varepsilon) - J_{12}^* J_{31}^* (\rho - \varepsilon).$$

The terms a and $a\prime$ and b and $b\prime$, respectively cancel each other. By inspection, it is easy to see that if $\varepsilon = \rho$, then (55) equals zero and therefore $\varepsilon = \rho$ is one of the eigenvalues of matrix (54), which is positive.

Further, notice that

[6] Please ask authors for detailed derivations if required.

$$H_1 + H_2 = -\frac{\left(Y_{1c}^* + \frac{1-\theta}{\theta} c_1^*\right)}{(p^*)^2} - \frac{\left(Y_{2c}^* + \frac{1-\theta}{\theta} c_2^*\right)}{(p^*)^2} = -\frac{C^*}{\theta (p^*)^2} \tag{56}$$

where $C^* = c_1^* + c_2^*$ and

$$\begin{aligned} \frac{\partial G(p^*, \lambda_2^*, K_1^*, K_2^*)}{\partial K_2} \quad &\equiv \quad G_{K_2}^* = -\frac{1}{(\mu+1)} \left(\frac{\lambda_2^*}{p^*}\right)^{\frac{1}{\theta}} \frac{(1-\alpha)^{\chi} (w_K^*)^{\chi}}{Q\gamma\chi\Omega} \\ &= \quad -\frac{1}{C^*} \frac{(1-\alpha)^{\chi} (w_K^*)^{\chi}}{Q\gamma\chi\Omega} \end{aligned} \tag{57}$$

and

$$\frac{\partial G(p^*, \lambda_2^*, K_1^*, K_2^*)}{\partial \lambda_2} \equiv G_{\lambda_2}^* = \frac{1}{\theta} \frac{1}{\lambda_2^*} \tag{58}$$

Using $(56) - (58)$, we obtain

$$\begin{aligned} -(H_1 + H_2) G_{K_2}^* \quad &= \quad \frac{C^*}{\theta (p^*)^2} G_{K_2}^* = -\frac{1}{\theta} \frac{(w_K^*)^{\chi}}{(p^*)^2} \frac{(1-\alpha)^{\chi}}{Q\gamma\chi\Omega} \\ &= \quad \Lambda G_{\lambda_2}^* \\ &\Rightarrow \quad \Lambda G_{\lambda_2}^* = \frac{C^*}{\theta (p^*)^2} G_{K_2}^* \end{aligned} \tag{59}$$

Using $(56) - (59)$, the Jacobian matrix (50) can be rewritten as

$$J^* = \begin{pmatrix} \frac{C^*}{\theta(p^*)^2} \left.\frac{\partial P}{\partial K_2}\right|_* & \Lambda \left.\frac{\partial P}{\partial K_2}\right|_* & \Lambda \left.\frac{\partial P}{\partial K_2}\right|_* \\ \frac{C^*}{\Lambda\theta(p^*)^2} \left(H_1 \left.\frac{\partial P}{\partial K_2}\right|_* + G_{K_2}^* \frac{c_1^*}{p^*} \right) & H_1 \left.\frac{\partial P}{\partial K_2}\right|_* + \rho & H_1 \left.\frac{\partial P}{\partial K_2}\right|_* \\ \frac{C^*}{\Lambda\theta(p^*)^2} \left(H_2 \left.\frac{\partial P}{\partial K_2}\right|_* + G_{K_2}^* \frac{c_2^*}{p^*} \right) & H_2 \left.\frac{\partial P}{\partial K_2}\right|_* & H_2 \left.\frac{\partial P}{\partial K_2}\right|_* + \rho \end{pmatrix} \tag{60}$$

Next we find the determinant of the Jacobian matrix (we first operate on matrix (54) to avoid notational cluster)

$$\begin{aligned}\det J^* &= J_{11}^*\left[\left(J_{23}^*+\rho\right)\left(J_{32}^*+\rho\right)-J_{23}^*J_{32}^*\right]-\underbrace{J_{12}^*J_{21}^*(J_{32}}_{a}+\rho)+\underbrace{J_{12}^*J_{23}^*J_{31}^*}_{b} \qquad (61)\\ &\quad +\underbrace{J_{12}^*J_{21}^*J_{32}^*}_{a'}-\underbrace{J_{12}^*J_{31}^*(J_{23}^*}_{b'}+\rho)\\ &= J_{11}^*\left[\left(J_{23}^*+\rho\right)\left(J_{32}^*+\rho\right)-J_{23}^*J_{32}^*\right]-J_{12}^*J_{21}^*\rho-J_{12}^*J_{31}^*\rho\\ &= \rho\left(J_{11}^*\left(J_{23}^*+J_{32}^*+\rho\right)-J_{12}^*\left(J_{21}^*+J_{31}^*\right)\right)\end{aligned}$$

Making appropriate substitutions from (60) and simplifying using (56) we obtain

$$\det J^* = \left.\frac{\partial P}{\partial K_2}\right|_* \frac{\rho C^*}{\theta\left(p^*\right)^2}\left(\rho-\frac{G_{K_2}}{p^*}C^*\right) \qquad (62)$$

Substituting for $G_{K_2}^*$ from (57) , we get

$$\begin{aligned}\det J^* &= \left.\frac{\partial P}{\partial K_2}\right|_* \frac{\rho C^*}{\theta\left(p^*\right)^2}\left(\rho-G_{K_2}^*\frac{C^*}{p^*}\right) \qquad (63)\\ &= -\frac{G_{K_2}^*}{G_p^*}\frac{\rho C^*}{\theta\left(p^*\right)^2}\left(\rho-G_{K_2}^*\frac{C^*}{w_K^*}r^*\right) \text{ multiplying and dividing by } w_K^* \text{ and simplifying}\\ &= -\frac{G_{K_2}^*}{G_p^*}\frac{\rho C^*}{\theta\left(p^*\right)^2}\left(\rho\left(1-G_{K_2}^*\frac{C^*}{w_K}\right)-G_{K_2}^*\frac{C^*}{w_K}\delta\right) \text{ substituing for } r^*=\delta+\rho\\ &= -\frac{G_{K_2}^*}{G_p^*}\frac{\rho C^*}{\theta\left(p^*\right)^2}\underbrace{\left(\rho\left(1+\frac{(1-\beta)^\chi}{\Omega}\left(\frac{r^*}{A}\right)^{\gamma\chi}-1\right)+\frac{(1-\beta)^\chi}{\Omega}\left(\frac{r^*}{A}\right)^{\gamma\chi}\delta-\delta\right)}_{\text{using (23) and (57)}}\\ &= \frac{1}{G_p^*}\frac{\rho}{\theta\left(p^*\right)^2}\frac{(1-\alpha)^\chi\left(w_K^*\right)^\chi}{Q^{\gamma\chi}\Omega^2}\left(\rho\left(1-\beta\right)^\chi\left(\frac{r^*}{A}\right)^{\gamma\chi}+\underbrace{\left(\left(1-\beta\right)^\chi\left(\frac{r^*}{A}\right)^{\gamma\chi}-\Omega\right)}_{\left(\frac{w_K^*}{Q}\right)^{\frac{\gamma}{1-\gamma}}(1-\alpha)^\chi \text{ from (23)}}\delta\right)\\ &= \frac{1}{G_p^*}\frac{\rho}{\theta\left(p^*\right)^2}\frac{(1-\alpha)^\chi\left(w_K^*\right)^\chi}{Q^{\gamma\chi}\Omega^2}\left(\rho\left(1-\beta\right)^\chi\left(\frac{r^*}{A}\right)^{\gamma\chi}+(1-\alpha)^\chi\delta\left(\frac{w_K^*}{Q}\right)^{\frac{\gamma}{1-\gamma}}\right)\end{aligned}$$

where G_p^* denotes the derivative of $G(p,\lambda_2,K_1,K_2)$ with respect to p from (35) evaluated at a steady state. The sign of $\det J^*$ therefore depends on the sign of G_p^*, which equals

$$\begin{aligned} G_p^* &= -\frac{1}{\theta}\frac{1}{p^*} - \frac{\chi}{C^*}\left(\frac{(1-\alpha)^{\chi}\left(w_K^*\right)^{\gamma\chi}\left.\frac{dW_K}{dp}\right|_{p=p^*}\left(K_1^*+K_2^*\right) - \alpha^{\chi}\left(w_L^{\gamma\chi}\right)\left.\frac{dW_L}{dp}\right|_{p=p^*}\left(L_1+L_2\right)}{Q^{\gamma\chi}\Omega}\right) \\ &= -\frac{1}{p^*}\left(\frac{1}{\theta} + \underbrace{\frac{\chi}{C^*}\frac{\left((1-\alpha)(1-\beta)\left(w_K^*\right)^{\gamma+1}\right)^{\chi}\left(K_1^*+K_2^*\right) + \left(\alpha\beta w_L^{\gamma+1}\right)^{\chi}\left(L_1+L_2\right)}{\left(Ap^*\right)^{\gamma\chi}Q^{\gamma\chi}\Omega^2}}_{>0 \text{ (using (14)}-\text{(15))}}\right) \end{aligned} \quad (64)$$

which is trivially negative. Since the determinant of a matrix equals the product of its eigenvalues, and since $\det J^* < 0$ and by (55) one of the eigenvalues of J^* equals $\rho > 0$, it must be the case that the other two eigenvalues have opposite signs.■

6.2. Analytical solution

To show that the model presented in section 2 with restrictions (45) generates a world's constant saving rate we first solve the closed economy Uzawa (1963) model of constant savings rates and then demonstrate that the model presented in section two and the model with constant savings rates are equivalent under the given restriction in parameter values.

Uzawa-Solow model closed economy.

We now solve the two-commodity closed economy model in which consumption and savings are a constant fraction of total income as in Uzawa (1963). Consider an economy with Cobb-Douglas production functions. To maximize profits, firms set

$$w_K^{Uz} = \alpha p^{Uz}\frac{Y_x^{Uz}}{K_x^{Uz}} = \beta\frac{Y_c^{Uz}}{K_c^{Uz}}, \qquad w_L^{Uz} = (1-\alpha)\,p^{Uz}\frac{Y_x^{Uz}}{L_x^{Uz}} = (1-\beta)\,\frac{Y_c^{Uz}}{L_c^{Uz}} \quad (65)$$

(we have dropped the country subscript as we are, at the moment, solving for the case of a closed economy). We use the superscript Uz to distinguish this model from the model presented in Section 2. Income Y^{Uz} equals labor income plus capital income, as follows

$$Y^{Uz} = w_L^{Uz}L^{Uz} + w_K^{Uz}K^{Uz} \quad (66)$$

where L^{Uz} and K^{Uz} are the labor and capital stock of a closed economy, respectively. Following Uzawa where a constant fraction of income is saved and a constant fraction of income is used for consumption, we have

$$C^{Uz} = (1-s)\,Y^{Uz}, \qquad sav^{Uz} = sY^{Uz} \quad (67)$$

where C^{Uz} denotes consumption, sav^{Uz} stand for savings and s is a constant satisfying $0 < s < 1$. We now set sav^{Uz} to be equal to the value of investment so that $sav^{Uz} = p^{Uz}Y_x^{Uz}$ and use the market clearing condition for the consumption good so that $C^{Uz} = Y_c^{Uz}$ equation (65) then becomes

$$w_K^{Uz} = \frac{\alpha s Y^{Uz}}{K_x^{Uz}} = \frac{\beta(1-s)Y^{Uz}}{K_c^{Uz}}, \qquad w_L^{Uz} = \frac{(1-\alpha)sY^{Uz}}{L_x^{Uz}} = \frac{(1-\beta)(1-s)Y^{Uz}}{L_c^{Uz}} \tag{68}$$

Let $0 < \nu < 1$ be the (possible variable) fraction of total capital used in the production of the investment good so that $K_x^{Uz} = \nu K^{Uz}$, and let η be the fraction of total labor employed in the production of the investment good equation (68) then becomes

$$w_K^{Uz} = \alpha\left(\frac{s}{\nu}\right)\frac{Y^{Uz}}{K^{Uz}} = \beta\left(\frac{1-s}{1-\nu}\right)\frac{Y^{Uz}}{K^{Uz}}, \tag{69}$$

$$w_L^{Uz} = (1-\alpha)\left(\frac{s}{\eta}\right)\frac{Y^{Uz}}{L^{Uz}} = (1-\beta)\left(\frac{1-s}{1-\eta}\right)\frac{Y^{Uz}}{L^{Uz}} \tag{70}$$

Since s is constant, (69) implies that v is also a constant equalling

$$\alpha\left(\frac{s}{\nu}\right) = \beta\left(\frac{1-s}{1-\nu}\right) \qquad \Rightarrow \qquad \nu = \frac{\alpha s}{\alpha s + \beta(1-s)} \tag{71}$$

Note that the larger the capital intensity of the investment good is, the larger the fraction of capital employed in the production of the investment good ν[7]. Similarly, η is also a constant equal to

$$(1-\alpha)\left(\frac{s}{\eta}\right) = (1-\beta)\left(\frac{1-s}{1-\eta}\right) \qquad \Rightarrow \qquad \eta = \frac{(1-\alpha)s}{(1-\alpha)s + (1-\beta)(1-s)}. \tag{72}$$

Since equation (71) indicates that the amount of capital K_x^{Uz} used in the production of the investment good is a constant fraction of total capital, using the equation of motion of capital, similar to that specified in (13), we get

$$\dot{K}^{Uz} = Y_x^{Uz} - \delta K^{Uz} = \left(\nu K^{Uz}\right)^{\alpha}(\eta L)^{1-\alpha} - \delta K^{Uz} \tag{73}$$

Note that (73) is a Bernoulli differential equation that can be solved analytically to obtain the path of capital K^{Uz} as follows

$$K^{Uz}(t) = \left(\left(K(0)^{1-\alpha} - \frac{A\nu^{\alpha}(\eta L)^{1-\alpha}}{\delta}\right)e^{-(1-\alpha)\delta t} + \frac{A\nu^{\alpha}(\eta L)^{1-\alpha}}{\delta}\right)^{\frac{1}{1-\alpha}} \tag{74}$$

[7] $\frac{\partial\nu}{\partial\alpha}\frac{\alpha}{\nu} = v\frac{\beta}{\alpha}\frac{1-s}{s}$

Taking the limit of (74) as t goes to infinity, one obtains the result that capital converges in the long run to

$$K^{Uz*} = \left(\frac{A\nu^{\alpha}}{\delta}\right)^{\frac{1}{1-\alpha}} \eta L \tag{75}$$

Using the market clearing condition for consumption, we obtain the consumption path

$$C^{Uz}(t) = Y_C^{Uz}(t) = Q\left((1-\nu)K^{Uz}(t)\right)^{\beta}\left((1-\eta)L\right)^{1-\beta} \tag{76}$$

and, therefore, consumption grows at the rate

$$\frac{\dot{C}^{Uz}}{C^{Uz}} = \beta\frac{\dot{K}^{Uz}}{K^{Uz}} \tag{77}$$

We now solve for the price of the investment good and labor wage rate in transition. Since $K_x^{Uz} = \nu K^{Uz}$, then

$$\frac{Y_x^{Uz}}{K_x^{Uz}} = \frac{A\left(\nu K^{Uz}\right)^{\alpha}(\eta L)^{1-\alpha}}{\nu K^{Uz}} \tag{78}$$

From the optimality conditions, it follows that

$$\alpha p^{Uz}\frac{Y_x^{Uz}}{K_x^{Uz}} = w_K^{Uz} \qquad \Rightarrow \qquad \alpha p^{Uz}\frac{A\left(\nu K^{Uz}\right)^{\alpha}(\eta L)^{1-\alpha}}{\nu K^{Uz}} = w_K^{Uz} \tag{79}$$

The price of the investment and the labor wage rate from (14) – (15) therefore equal

$$p^{Uz}(t) = \frac{\bar{Q}}{\bar{A}}\left(\frac{\alpha}{1-\alpha}\frac{\eta L}{\nu K^{Uz}(t)}\right)^{\alpha-\beta}, \tag{80}$$

$$w_L^{Uz}(t) = \bar{Q}\left(\frac{1-\alpha}{\alpha}\frac{\nu K^{Uz}(t)}{\eta L}\right)^{\beta}$$

Substituting (80) into (79), we obtain the path of w_K which is given by

$$w_K^{Uz} = \bar{Q}\left(\frac{\alpha}{1-\alpha}\frac{\eta}{\nu}\frac{L}{K(t)^{Uz}}\right)^{1-\beta} \tag{81}$$

Ramsey-Uzawa-Solow equivalence.

We now show that the Uzawa-Solow two-sector model and the two-sector model with endogenous savings rates are equivalent when

$$\rho = \delta\left(\alpha + \beta\left(\theta - 1\right)\right) - \delta, \tag{82}$$

the restriction

$$\alpha + \beta\left(\theta - 1\right) > 1 \tag{83}$$

holds, and $s = \frac{1}{\theta}$ (where we require $\theta > 1$). Setting $s = \frac{1}{\theta}$, v (from (71)) equals

$$v = \frac{\alpha}{\alpha + \beta\left(\theta - 1\right)} \tag{84}$$

With the superscript Uz we denote the derivations that come from the Uzawa-Solow type model. From (69) , income Y^{Uz} equals $\frac{\nu}{s}\frac{w_K^{Uz}}{\alpha}K^{Uz}$. Hence, $\frac{\dot{Y}^{Uz}}{Y^{Uz}} = \frac{\dot{w}_K^{Uz}}{w_K^{Uz}} + \frac{\dot{K}^{Uz}}{K^{Uz}}$. Given that $C^{Uz} = (1 - s)\,Y^{Uz}$, the following rates of change are equal

$$\frac{\dot{C}^{Uz}}{C^{Uz}} = \frac{\dot{Y}^{Uz}}{Y^{Uz}} = \frac{\dot{w}_K^{Uz}}{w_K^{Uz}} + \frac{\dot{K}^{Uz}}{K^{Uz}} \tag{85}$$

Since profits are maximized under the Uzawa-Solow setting, equation (14) also holds and therefore

$$\frac{\dot{w}_K^{Uz}}{w_K^{Uz}} = -\left(\frac{1-\beta}{\beta-\alpha}\right)\frac{\dot{p}^{Uz}}{p^{Uz}}. \tag{86}$$

Substituting (86) into (85) , we obtain

$$\frac{\dot{C}^{Uz}}{C^{Uz}} = -\left(\frac{1-\beta}{\beta-\alpha}\right)\frac{\dot{p}^{Uz}}{p^{Uz}} + \frac{\dot{K}^{Uz}}{K^{Uz}}. \tag{87}$$

From (80) it follows that

$$\frac{\dot{p}^{Uz}}{p^{Uz}} = (\beta - \alpha)\,\frac{\dot{K}^{Uz}}{K^{Uz}} \tag{88}$$

Using (87), (88) and adding and subtracting $\frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}}$, we get

$$\frac{\dot{C}^{Uz}}{C^{Uz}} = \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} - \frac{\dot{p}^{Uz}}{p^{Uz}}\left(\frac{1}{\theta} + \left(\frac{1-\beta}{\beta-\alpha}\right)\right) + \frac{\dot{K}^{Uz}}{K^{Uz}} \tag{89}$$

$$= \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} - (\beta-\alpha)\frac{\dot{K}^{Uz}}{K^{Uz}}\left(\frac{1}{\theta} + \left(\frac{1-\beta}{\beta-\alpha}\right)\right) + \frac{\dot{K}^{Uz}}{K^{Uz}}$$

$$= \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} + \left(\frac{\theta-(\beta-\alpha)-\theta(1-\beta)}{\theta}\right)\frac{\dot{K}^{Uz}}{K^{Uz}}$$

$$= \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} + \left(\frac{\alpha+\beta(\theta-1)}{\theta}\right)\frac{\dot{K}^{Uz}}{K^{Uz}}$$

$$= \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} + \left(\frac{\alpha+\beta(\theta-1)}{\theta}\right)\left(\frac{Y_x^{Uz}-\delta K^{Uz}}{K^{Uz}}\right) \text{ using } (73) \tag{90}$$

From (69) we obtain $\frac{Y_x^{Uz}}{K^{Uz}} = \frac{w_K^{Uz}}{p^{Uz}}\frac{\nu}{\alpha}$, and substituting this result into (90) yields

$$\frac{\dot{C}^{Uz}}{C^{Uz}} = \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} + \left(\frac{\alpha+\beta(\theta-1)}{\theta}\right)\left(\frac{w_K^{Uz}}{p^{Uz}}\frac{\nu}{\alpha} - \delta\right) \tag{91}$$

$$= \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} + \frac{\nu}{\alpha}\left(\frac{\alpha+\beta(\theta-1)}{\theta}\right)\frac{w_K^{Uz}}{p^{Uz}} - \left(\frac{\alpha+\beta(\theta-1)}{\theta}\right)\delta$$

In the case of endogenous savings, (the model presented in section 2), the consumer Euler condition is given by (5)

$$\frac{\dot{C}^{R}}{C^{R}} = \frac{1}{\theta}\frac{\dot{p}^{R}}{p^{R}} + \frac{1}{\theta}\frac{w_K^{R}}{p^{R}} - \frac{\delta+\rho}{\theta} \tag{92}$$

where the superscript R (Ramsey type model) is used to distinguish variables that come from the model when savings rates are endogenous. Now, using

$$\rho = \delta\left(\alpha+\beta(\theta-1)\right) - \delta \tag{93}$$

from (45) and from (84), it follows that

$$v = \frac{\alpha}{\alpha+\beta(\theta-1)} \quad \text{or} \quad \frac{1}{\theta} = \frac{v}{\alpha}\left(\frac{\alpha+\beta(\theta-1)}{\theta}\right) \tag{94}$$

to obtain

$$\begin{aligned}\frac{\dot{C}^R}{C^R} &= \frac{1}{\theta}\frac{\dot{p}^R}{p^R}+\frac{1}{\theta}\frac{w_K^R}{p^R}-\left(\frac{(\alpha+\beta(\theta-1))}{\theta}\right)\delta \\ &= \frac{1}{\theta}\frac{\dot{p}^R}{p^R}+\frac{v}{\alpha}\left(\frac{\alpha+\beta(\theta-1)}{\theta}\right)\frac{w_K^R}{p^R}-\left(\frac{(\alpha+\beta(\theta-1))}{\theta}\right)\delta \\ &= \frac{\dot{C}^{Uz}}{C^{Uz}}\ \blacksquare\end{aligned} \tag{95}$$

6.3. *Proof of proposition 5*

Let $\hat{\lambda}_j=\lambda_j e^{\rho t}$. Taking the log time derivative of $\hat{\lambda}_j$ and employing (4), we get

$$\frac{\dot{\hat{\lambda}}_j}{\hat{\lambda}_j}=\frac{\dot{\lambda}_j}{\lambda_j}-\rho=-\frac{w_K}{p}+\delta. \tag{96}$$

Integrating the consumers' budget and using $c_j^{-\theta}p=\hat{\lambda}_j e^{\rho t}$ (from (4)) and setting $c_j(t)=\phi_j C(t)$, we obtain

$$\hat{\lambda}_j(t)K_j = \hat{\lambda}_j(0)K_j(0)+\phi_j^{-\theta}\int_0^t w_L(\tau)C(\tau)^{-\theta}e^{-\rho\tau}d\tau-\phi_j^{1-\theta}\int_0^t C(\tau)^{1-\theta}e^{-\rho\tau}d\tau \tag{97}$$

To obtain ϕ_j, we first solve for the integrals in equation (97). Note that since the world (aggregate) economy behaves as a closed economy we can employ the results from the analytical solution for the Uzawa-Solow type model. We use (76) and (80) to obtain

$$w_L C^{-\theta}=H^{-\theta}\bar{Q}\left(\frac{1-\alpha}{\alpha}\frac{\nu}{\eta L}\right)^{\beta}\frac{K^{\beta(1-\theta)}}{L^{\theta(1-\beta)}} \tag{98}$$

where $H=Q(1-\nu)^{\beta}(1-\eta)^{1-\beta}$, and

$$C^{1-\theta}=H^{1-\theta}K^{\beta(1-\theta)}L^{(1-\beta)(1-\theta)} \tag{99}$$

Substituting (98) and (99) into (97), we get

$$\hat{\lambda}_j(t)K_j = \hat{\lambda}_j(0)K_j(0)+\left(\phi_j^{-\theta}\left(\frac{1-\alpha}{\alpha}\frac{\nu}{\eta L}\right)^{\beta}\frac{H^{-\theta}\bar{Q}}{L^{\theta(1-\beta)}}-\left(\phi_j H L^{1-\beta}\right)^{1-\theta}\right)\int_0^t K(\tau)^{\beta(1-\theta)}e^{-\rho\tau}d\tau \tag{100}$$

Using (74) and (45) and integrating $\int_0^t K(\tau)^{\beta(1-\theta)} e^{-\rho\tau} d\tau$, we get

$$\begin{aligned}
&\int_0^t K(\tau)^{\beta(1-\theta)} e^{-\rho\tau} d\tau \\
= \;& \int_0^t \underbrace{\left(\left(\frac{\left(K(0)^{1-\alpha} - \left(K^{Uz*}\right)^{1-\alpha}\right)}{e^{(1-\alpha)\delta\tau}} + \left(K^{Uz*}\right)^{1-\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{\beta(1-\theta)}}_{K(\tau) \text{ from } (74)} e^{-\rho\tau} dt \\
= \;& \int_0^t \left(K(0)^{1-\alpha} - \left(K^{Uz*}\right)^{1-\alpha} + \left(K^{Uz*}\right)^{1-\alpha} e^{(1-\alpha)\delta\tau}\right)^{\frac{\beta(1-\theta)}{1-\alpha}} * \\
& * e^{\overbrace{(-\delta(\alpha+\beta(\theta-1))+\delta)}^{-\rho \text{ from } (45)}\tau - \delta\beta(1-\theta)\tau} d\tau \\
= \;& \int_0^t \left(K(0)^{1-\alpha} - \left(K^{Uz*}\right)^{1-\alpha} + \left(K^{Uz*}\right)^{1-\alpha} e^{(1-\alpha)\delta\tau}\right)^{-\frac{\rho}{(1-\alpha)\delta}-1} e^{(1-\alpha)\delta\tau} d\tau \\
= \;& -\left.\frac{\left(K(0)^{1-\alpha} - \left(K^{Uz*}\right)^{1-\alpha} + \left(K^{Uz*}\right)^{1-\alpha} e^{(1-\alpha)\delta\tau}\right)^{-\frac{\rho}{(1-\alpha)\delta}}}{\left(K^{Uz*}\right)^{1-\alpha}\rho}\right|_0^t \\
= \;& \frac{1}{\left(K^{Uz*}\right)^{1-\alpha}\rho}\left(\frac{1}{K(0)^{\frac{\rho}{\delta}}} - \frac{1}{\left(K(t)^{Uz}\right)^{\frac{\rho}{\delta}} e^{\rho t}}\right) \text{ using } (74). \qquad (101)
\end{aligned}$$

Substituting (101) into (100) yields

$$\begin{aligned}
\hat{\lambda}_j(t) K_j(t) &= \hat{\lambda}_j(0) K_j(0) + \qquad (102) \\
&+ \left(\bar{Q}\left(\frac{1-\alpha}{\alpha}\frac{\nu}{\eta L}\right)^{\beta} - \phi_j H L^{(1-\beta)}\right) \frac{\phi_j^{-\theta}}{H^{\theta} L^{\theta(1-\beta)} \rho \left(K^{Uz*}\right)^{1-\alpha}} \left(\frac{1}{K(0)^{\frac{\rho}{\delta}}} - \frac{1}{K(t)^{\frac{\rho}{\delta}} e^{\rho t}}\right)
\end{aligned}$$

Using $c_j(t)^{-\theta} p(t) e^{-\rho t} = \hat{\lambda}_j(t)$ from (4) and setting $c_j(t) = \phi_j C(t)$, we get

$$\begin{aligned}
\hat{\lambda}_j(t) K_j(t) &= \phi_j^{-\theta} C(t)^{-\theta} p(t) e^{-\rho t}(t) K_j(t) \qquad (103)\\
&= \phi_j^{-\theta} C(0)^{-\theta} p(0) K_j(0) + \\
&\quad + \left(\bar{Q}\left(\frac{1-\alpha}{\alpha}\frac{\nu}{\eta L}\right)^{\beta} - \phi_j H L^{(1-\beta)}\right) \frac{\phi_j^{-\theta}}{H^{\theta} L^{\theta(1-\beta)} \rho \left(K^{Uz*}\right)^{1-\alpha}} \left(\frac{1}{K(0)^{\frac{\rho}{\delta}}} - \frac{1}{K(t)^{\frac{\rho}{\delta}} e^{\rho t}}\right).
\end{aligned}$$

Taking the limit as t approaches infinity and using the transversality condition, we obtain

$$0 = C(0)^{-\theta} p(0) K_j(0) + \frac{\left(\bar{Q}\left(\frac{1-\alpha}{\alpha}\frac{\nu}{\eta L}\right)^{\beta} - \phi_j H L^{(1-\beta)}\right)}{H^{\theta} L^{\theta(1-\beta)} \rho \left(K^{Uz*}\right)^{1-\alpha} K(0)^{\frac{\rho}{\delta}}}. \qquad (104)$$

Using the analytical solution for C and p (from (76) and (80)) yields

$$C(0)^{-\theta} p(0) = \frac{K(0)^{\beta(1-\theta)-\alpha}}{H^{\theta} L^{\theta(1-\beta)}} \frac{\bar{Q}}{\bar{A}} \left(\frac{\alpha}{1-\alpha}\frac{\eta L}{\nu}\right)^{\alpha-\beta} \qquad (105)$$

substituting into (104), using $\frac{\rho}{\delta} = (\alpha + \beta(\theta - 1)) - 1$ (from (45)), setting $K^{Uz*} = \left(\frac{A\nu^{\alpha}}{\delta}\right)^{\frac{1}{1-\alpha}} \eta L$ (from (75)) and solving for ϕ_j, we get

$$\begin{aligned}
\phi_j &= \frac{\bar{Q}}{HL}\left(\frac{1-\alpha}{\alpha}\frac{\nu}{\eta}\right)^{\beta}\left(1 + \frac{\rho}{\bar{A}}\left(K^{Uz*}\right)^{1-\alpha} K_j(0) K(0)^{\frac{\rho}{\delta}+\beta(1-\theta)-\alpha}\left(\frac{\alpha}{1-\alpha}\frac{\eta L}{\nu}\right)^{\alpha}\right)\\
&= \frac{\beta^{\beta}(1-\beta)^{1-\beta}}{(1-\nu)^{\beta}(1-\eta)^{1-\beta}}\left(\frac{\nu}{\eta}\frac{1-\alpha}{\alpha}\right)^{\beta}\left(\frac{\rho}{\delta}\frac{\eta}{1-\alpha}\frac{K_j(0)}{K(0)} + \frac{1}{L}\right) \qquad (106)
\end{aligned}$$

Adding ϕ_j for $j = 1, 2$, leads to the result

$$\phi_1 + \phi_2 = 1 = \frac{\beta^{\beta}(1-\beta)^{1-\beta}}{(1-\nu)^{\beta}(1-\eta)^{1-\beta}}\left(\frac{\nu}{\eta}\frac{1-\alpha}{\alpha}\right)^{\beta}\left(\frac{\rho}{\delta}\frac{\eta}{1-\alpha} + 1\right) \qquad (107)$$

which implies

$$\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{(1-\nu)^{\beta}(1-\eta)^{1-\beta}}\left(\frac{\nu}{\eta}\frac{1-\alpha}{\alpha}\right)^{\beta} = \frac{\delta(1-\alpha)}{\rho\eta + \delta(1-\alpha)} \qquad (108)$$

From (45), we obtain

$$-\frac{\rho}{\delta}\frac{1}{(\theta-1)} = \frac{1-\alpha}{\theta-1}-\beta, \tag{109}$$

Using $s=\frac{1}{\theta}$, η, as defined in (72), η therefore equals

$$\eta = \frac{(1-\alpha)}{(1-\alpha)+(1-\beta)(\theta-1)} \tag{110}$$

Substituting for η into $\frac{\delta(1-\alpha)}{\rho\eta+\delta(1-\alpha)}$, we get

$$\frac{\delta(1-\alpha)}{\rho\eta+\delta(1-\alpha)} = \frac{(1-\alpha)+(1-\beta)(\theta-1)}{(\theta-1)} \tag{111}$$

thus (106) becomes

$$\phi_j = \underbrace{\frac{\delta(1-\alpha)}{\rho\eta+\delta(1-\alpha)}}_{\text{using (108)}}\left(\frac{\rho}{\delta}\frac{\eta}{1-\alpha}\frac{K_j(0)}{K(0)}+\frac{1}{L}\right) \tag{112}$$

$$= \underbrace{\frac{(1-\alpha)+(1-\beta)(\theta-1)}{(\theta-1)}}_{\text{using (111)}}\left(\frac{\rho}{(1-\alpha)\delta}\underbrace{\frac{(1-\alpha)}{(1-\alpha)+(1-\beta)(\theta-1)}}_{\text{using (110)}}\frac{K_j(0)}{K(0)}+\frac{1}{L}\right)$$

$$= \frac{\rho}{\delta}\frac{1}{(\theta-1)}\frac{K_j(0)}{K(0)}+\frac{(1-\alpha)+(1-\beta)(\theta-1)}{(\theta-1)}\frac{1}{L}$$

$$= \frac{\rho}{\delta}\frac{1}{(\theta-1)}\left(\frac{K_j(0)}{K(0)}-\frac{1}{L}\right)+\frac{1}{L}\text{ using (109)} \quad \blacksquare$$

6.4. ***Proof of proposition 6***

Let $R(t)$ denote the net returns to capital at instant t, which equal the returns from renting out a unit of capital $\frac{w_K(t)}{p(t)}$ plus capital gains $\frac{\dot{p}(t)}{p(t)}$ minus the value loss of capital due to depreciation

δ. From (14), (74) and (80) it follows that the net returns to capital with trade at instant t equal

$$R^T(t) = \frac{w_K^T(t)}{p^T(t)} + \frac{\dot{p}^T(t)}{p^T(t)} - \delta = \frac{w_K^T(t)}{p^T(t)} \underbrace{- \frac{\dot{K}^T(t)}{K^T(t)}(\alpha-\beta)}_{\text{from (80)}} - \delta \tag{113}$$

$$= \underbrace{\alpha A \left(\frac{\eta L}{\nu K^T(t)}\right)^{1-\alpha}}_{\text{from (79)}} - \underbrace{\left(A\nu^\alpha \left(\frac{\eta L}{K^T(t)}\right)^{1-\alpha} - \delta\right)}_{\text{from (73)}} (\alpha-\beta) - \delta$$

$$= \alpha A \left(\frac{\eta}{\nu}\frac{L}{K^T(t)}\right)^{1-\alpha} - \underbrace{\left(\left(\frac{A\nu^\alpha}{\delta}\left(\frac{\eta L}{K^T(t)}\right)^{1-\alpha} - 1\right)(\alpha-\beta) + 1\right)\delta}_{\text{Factoring } \delta}$$

We use the superscript T to indicate the trade scenario. As before, $L = L_1 + L_2$ holds and $K^T = K_1^T + K_2^T$ is the world capital stock under trade. In autarchy (denoted by superscript A), capital in country j evolves, similar to (74), as follows

$$K_j^A(t) = \left(\left(K_j(0)^{1-\alpha} - \frac{A\nu^\alpha(\eta L_j)^{1-\alpha}}{\delta}\right)e^{-(1-\alpha)\delta t} + \frac{A\nu^\alpha(\eta L_j)^{1-\alpha}}{\delta}\right)^{\frac{1}{1-\alpha}} \tag{114}$$

The ratio between the rental price of capital and the price of the investment good of country j in autarchy $\left(\frac{w_{Kj}^A}{p_j^A}\right)$ equals

$$\frac{w_{Kj}^A(t)}{p_j^A(t)} = \alpha A \left(\frac{\eta L_j}{\nu K_j^A(t)}\right)^{1-\alpha} \tag{115}$$

Let $\frac{K_j(0)}{L_j} > \frac{K_i(0)}{L_i}$. In autarchy, the net returns to the capital of country j equal $R_j^A(t)$

$$R_j^A(t) = \frac{w_{Kj}^A(t)}{p_j^A(t)} + \frac{\dot{p}_j^A(t)}{p_j^A(t)} - \delta \tag{116}$$

$$= \alpha \left(\frac{\eta}{\nu}\frac{L_j}{K_j^A(t)}\right)^{1-\alpha} - \left(\left(\left(\frac{\nu^\alpha}{\delta}\right)\left(\frac{\eta L_j}{K_j^A(t)}\right)^{1-\alpha} - 1\right)(\alpha-\beta) + 1\right)\delta$$

A similar expression holds for country i. From (114) one can readily verify that in autarchy, if country j has a larger initial endowment of capital than country $i \neq j$, then country j will always have a larger capital stock and both countries reach the same level of capital asymptotically as time approaches infinity. Thus, since $\frac{K_j(0)}{L_j} > \frac{K_i(0)}{L_i}$ implies $\frac{K_j^A(t)}{L_j} > \frac{K_i^A(t)}{L_i}$ and from (116) the effect of per-capita capital of country j on R_j^A equals (using (84)),

$$(1-\alpha)\,\eta\left(\frac{\nu}{\eta}\right)^{\alpha}\left(\frac{L_j}{K_j^A(t)}\right)^{(1-\alpha)}\left(\frac{L_j}{K_j^A(t)}\right)\underbrace{\left(-\frac{\alpha}{\nu}+(\alpha-\beta)\right)}_{-\beta\theta=\text{ from}(84)} \tag{117}$$

$$= \quad -(1-\alpha)\,\eta\left(\frac{\nu}{\eta}\right)^{\alpha}\left(\frac{L_j}{K_j^A(t)}\right)^{(1-\alpha)}\left(\frac{L_j}{K_j^A(t)}\right)\beta\theta<0$$

(a similar expression holds for country i), then $R_i^A(t)>R_j^A(t)$. Note that in the case of trade, the effect of world per-capita capital on R^T equals

$$-(1-\alpha)\,\eta\left(\frac{\nu}{\eta}\right)^{\alpha}\left(\frac{L}{K^T(t)}\right)^{(1-\alpha)}\left(\frac{L}{K^T(t)}\right)\beta\theta<0 \tag{118}$$

It only remains to show that $R_i^A(t)>R^T(t)>R_j^A(t)$. To show this from (113) and (116) it suffices to show that $\frac{K_j^A(t)}{L_j}>\frac{K^T(t)}{L(t)}=\frac{K_1^T(t)+K_2^T(t)}{L_1+L_2}>\frac{K_i^A(t)}{L_i}$. Since $\frac{K_j(0)}{L_j}>\frac{K_1(0)+K_2(0)}{L_1+L_2}>\frac{K_i(0)}{L_i}$ always holds, using (74) and (114) it follows that $\frac{K_j^A(t)}{L_j}>\frac{K^T(t)}{L(t)}=\frac{K_1^T(t)+K_2^T(t)}{L_1+L_2}>\frac{K_i^A(t)}{L_i}$ also holds and, therefore, $R_i^A(t)>R^T(t)>R_j^A(t)$ $\blacksquare$

6.5. ***Proof of proposition 7***

Let $\frac{K_j(0)}{L_j}>\frac{K_i(0)}{L_i}$. Using (80), in autarchy the labor wage rate of country j equals

$$w_j^A(t)=\bar{Q}\left(\frac{1-\alpha}{\alpha}\frac{\nu K_j^A(t)}{\eta L_j}\right)^{\beta}, \tag{119}$$

and a similar expression holds for country $i\neq j$. With trade, the labor wage rate equals

$$w^T(t)=\bar{Q}\left(\frac{1-\alpha}{\alpha}\frac{\nu K^T(t)}{\eta L}\right)^{\beta} \tag{120}$$

since $\frac{K_j^A(t)}{L_j}>\frac{K^T(t)}{L(t)}=\frac{K_1^T(t)+K_2^T(t)}{L_1+L_2}>\frac{K_i^A(t)}{L_i}$ it follows that $w_j^A(t)>w^T(t)>w_i^A(t)$.

6.6. ***Proof of proposition 8***

Using (103) it can also be verified that

$$-\hat{\lambda}_j(t)\,K_j(t) \quad = \quad \int_t^{\infty}\left(w_L(\tau)\,c_j(\tau)^{-\theta}-c_j(\tau)^{1-\theta}\right)e^{-\rho\tau}d\tau \tag{121}$$

$$= \quad \left(\left(\frac{1-\alpha}{\alpha}\frac{\nu}{\eta L}\right)^{\beta}\frac{\phi_j^{-\theta}\bar{Q}}{H^{\theta}L^{(1-\beta)\theta}}-\frac{\phi_j^{1-\theta}}{H^{-1+\theta}L^{(1-\beta)(\theta-1)}}\right)\int_t^{\infty}K(\tau)^{\beta(1-\theta)}e^{-\rho\tau}d\tau$$

or integrating using (74)

$$\underbrace{p(t)\,C(t)^{-\theta}\,e^{-\rho t}}_{\hat{\lambda}_j(t)/\phi_j^{-\theta}}K_j = \left(\frac{HL^{(1-\beta)}\phi_j - \bar{Q}\left(\frac{1-\alpha}{\alpha}\frac{\nu}{\eta L}\right)^{\beta}}{\left(K^{Uz*}\right)^{1-\alpha}\rho K(t)^{\frac{\rho}{\delta}}\,e^{\rho t}}\right)\frac{1}{L^{(1-\beta)\theta}H^{\theta}} \tag{122}$$

Using (45), (76), (80) and (75) simplification leads to[8]

$$\kappa_j = \frac{K_j(t)}{K(t)} = \frac{\delta}{\rho}\frac{1-\alpha}{\eta}\left(\phi_j\left(\frac{1-\eta}{1-\beta}\right) - \frac{1}{L}\right) \tag{123}$$

Hence, the ratio $\frac{K_j(t)}{K(t)}$ is constant for all t. Solving for ϕ_j, we obtain

$$\phi_j = \left(\frac{\rho}{\delta}\frac{\eta}{1-\alpha}\kappa_j + \frac{1}{L}\right)\frac{1-\beta}{1-\eta} \tag{124}$$

$$= \left(\frac{\rho}{\delta}\frac{1}{\theta-1}\right)\left(\kappa_j(0) - \frac{1}{2}\right) + \frac{1}{2} \;\blacksquare \tag{125}$$

6.7. ***Proof of proposition 9***

Per country income is given by

$$Y_j = w_L L_j + w_K K_j \tag{126}$$

$$= \bar{Q}\left(L_j + \frac{\alpha}{1-\alpha}\frac{\eta}{\nu}L\frac{K_j(t)}{K(t)}\right)\left(\frac{1-\alpha}{\alpha}\frac{\nu}{\eta}\frac{K(t)}{L}\right)^{\beta}$$

$$= \bar{Q}\left(\frac{1-\alpha}{\alpha}\frac{\nu}{\eta}\right)^{\beta}\left(\frac{L_j}{L} + \frac{\alpha}{1-\alpha}\frac{\eta}{\nu}\kappa_j\right)K(t)^{\beta}L^{1-\beta}$$

Since the ratio $\frac{K_j(t)}{K(t)}$ is constant, this implies that income in each country grows at the same rate as aggregate (world) capital which, in turn, implies

$$\frac{\dot{Y}_j}{Y_j} = \beta\frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{c}_j}{c_j} \tag{127}$$

Since the consumption and income of country j grow at the same rate, the ratio $\frac{c_j}{Y_j} \equiv (1-s_j)$ is constant, that is, the consumption expenditure is a constant fraction of income. Using $1 =$

[8] Notice that from (71) – (72)

$$\frac{\alpha}{\beta}\left(\frac{1-\beta}{1-\alpha}\right) = \frac{\nu}{\eta}\left(\frac{1-\eta}{1-\nu}\right)$$

$\frac{\beta}{\alpha}\frac{1-\alpha}{1-\beta}\frac{\nu}{\eta}\frac{1-\eta}{1-\nu}$ and $H = Q(1-\nu)^{\beta}(1-\eta)^{1-\beta}$ it can be shown that the consumption share of country j equals

$$\begin{aligned} 1-s_j &= \frac{c_j}{Y_j} = \frac{\phi_j C_j}{Y_j} = \frac{(1-\beta)^{\beta}}{\beta^{\beta}}\frac{(1-\nu)^{\beta}}{(1-\eta)^{\beta}}\frac{\alpha^{\beta}}{(1-\alpha)^{\beta}}\frac{\eta^{\beta}}{\nu^{\beta}}\frac{\phi_j(1-\eta)K^{Uz}(t)^{\beta}L^{1-\beta}}{(1-\beta)\left(L_j+\frac{\alpha}{1-\alpha}\frac{\eta}{\nu}L\kappa_j\right)\left(\frac{K(t)}{L}\right)^{\beta}} \\ &= \frac{(1-\eta)}{(1-\beta)}\frac{\phi_j}{\left(\frac{L_j}{L}+\frac{\alpha}{1-\alpha}\frac{\eta}{\nu}\kappa_j\right)} \\ &= \frac{\left(\frac{L_j}{L}+\frac{\rho}{\delta}\frac{\eta}{1-\alpha}\kappa_j\right)}{\left(\frac{L_j}{L}+\frac{\alpha}{1-\alpha}\frac{\eta}{\nu}\kappa_j\right)} \end{aligned} \tag{128}$$

which implies

$$\begin{aligned} s_j &= \frac{\eta}{1-\alpha}\frac{\frac{\alpha}{\nu}-\frac{\rho}{\delta}}{\left(\frac{L_j}{L}\frac{K(0)}{K_j(0)}+\frac{\alpha}{1-\alpha}\frac{\eta}{\nu}\right)} \\ &= \frac{\eta}{1-\alpha}\frac{1}{\left(\frac{L_j}{L}\frac{K(0)}{K_j(0)}+\frac{\eta}{1-\alpha}\frac{\delta+\rho}{\delta}\right)} \\ &= \frac{\eta}{1-\alpha}\frac{1}{\left(\frac{L_j}{L}\frac{K(0)}{K_j(0)}+\frac{\delta+\rho}{\delta(\theta-1)-\rho}\right)} \\ &= \frac{\delta}{\frac{L_j}{L}\frac{K(0)}{K_j(0)}(\delta(\theta-1)-\rho)+\delta+\rho} \\ &= \frac{\delta}{[\delta((1-\alpha)+(1-\beta)(\theta-1))]\left(\frac{L_j}{L}\frac{K(0)}{K_j(0)}\right)+\delta+\rho} \end{aligned} \tag{129}$$

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